SHIMURA- AND TEICHMÜLLER CURVES

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Abstract. We classify curves in the moduli space of curves M_g that are both Shimura- and Teichmüller curves: For both $g = 3$ and $g = 4$ there exists precisely one such curve, for $g = 2$ and $g \geq 6$ there are no such curves.
We start with a Hodge-theoretic description of Shimura curves and of Teichmüller curves that reveals similarities and differences of the two classes of curves. The proof of the classification relies on the geometry of square-tiled coverings and on estimating the numerical invariants of these particular fibered surfaces.
Finally we translate our main result into a classification of Teichmüller curves with totally degenerate Lyapunov spectrum.

Introduction

A Teichmüller curve is an algebraic curve $C \to M_g$ in the moduli space of curves, which is the image of a holomorphic geodesic for the Teichmüller (equivalently: Kobayashi-) metric $\tilde{\jmath} : \mathbb{H} \to T_g$ in Teichmüller space. Only rarely geodesics in Teichmüller space map to algebraic curves in M_g.

By Teichmüller's theorems a geodesic $\tilde{\jmath}$ is generated by a Riemann surface X with a Teichmüller marking together with a quadratic differential q on X. By [Kr81] (or [McM03] Theorem. 4.1) the composition $C \to M_g \to A_g$ is a geodesic for the Kobayashi metric precisely if $q = \omega^2$ for some holomorphic 1-form ω on X. We deal here exclusively with these Teichmüller curves.

A Shimura curve of Hodge type is a curve $\mathbb{H}/\Gamma =: C \to A_g$ in the moduli space of abelian varieties of dimension g that is totally geodesic for the Hodge metric on A_g and that contains a CM point. Such curves are automatically algebraic. See Section 1 for a group-theoretic definition of Shimura curves.

Both Shimura curves and Teichmüller curves can be characterized by their variation of Hodge structures (VHS). See Section 1 and Section 2 for similarities and differences between the two sorts of curves. One also could take the Theorems 1.2 and 2.2 stated there as definition of Shimura- resp. Teichmüller curves.

Here we investigate whether there are curves with *both* properties, i.e. curves in M_g that are Teichmüller curves and, when considered in A_g, Shimura curves. We call them *ST-curves* for short. The motivation for studying them is threefold:
First, the VHS of a Teichmüller curve consists of a sub-local system $\mathbb{L}$ that is maximal Higgs (see Section 2 for the definition), its conjugates and some rest $\mathbb{M}$. Not

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much is known about $\mathbb{M}$, except that $\mathbb{M}$ is a 'contraction' of $\mathbb{L}$ (see [McM03]) in the following sense: If $\gamma \in \pi_1(C)$ is hyperbolic, the largest eigenvalue of γ acting on a fiber of $\mathbb{L}$ is strictly bigger than the the largest eigenvalue of γ acting on a fiber of $\mathbb{M}$. Shimura curves correspond to the case where $\mathbb{M}$ is as 'small' as possible, namely trivial.
Second, they are curves with 'few' singular fibers: Beauville and Tan have shown that semistable fibrations over $\mathbb{P}^1$ have at least 4 singular fibers resp. 5, if the fiber genus is at least two. In [TaTuZa05] the authors study semistable fibrations over $\mathbb{P}^1$ with 5 and 6 singular fibers. Instead of sticking to the basis $\mathbb{P}^1$ one could admit any base and ask for the following: Classify fibrations that have few singular fibers with proper Jacobian compared to the number of singular fibers with non-smooth fibrations! The curves that are both Shimura- and Teichmüller- have *no* singular fibers with smooth Jacobian by Proposition 2.4.
The third motivation concerns the Lyapunov exponents of the of lift of the Teichmüller geodesic flow to the Hodge bundle (see Section 5). For the natural measure supported on whole strata of the Hodge bundle the spectrum of Lyapunov exponents has recently been shown to be simple, i.e. the Lyapunov exponents are distinct ([Fo02], [AV07]). In contrast, the Lyapunov spectrum for the natural measure supported on an individual Teichmüller curves can be maximally degenerate, i.e. all but the top and bottom Lyapunov exponent are zero. We will show in Section 5 that these curves are precisely the ST-curves.

As explained in Section 2 we abuse the notion Teichmüller curve for étale coverings of what we have defined above. Our main result is:

Theorem 4.1: *For $g = 2$ and $g \geq 6$ there are no ST-curves.*
In both M_3 and in M_4 there is only one ST-curve. Its universal family is given by

$$y^4 = x(x-1)(x-t)$$

in M_3 and respectively in M_4 by

$$y^6 = x(x-1)(x-t).$$

In both cases $t \in \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$.

We remark that the combinatorial discussion in the cases $g = 4$ and $g = 5$ was incomplete when earlier versions of this paper circulated. The example in M_4 was shown to the author by Forni and Matheus ([FoMa08]). There are many constraints for the existence of a ST-curve in genus 5, see Corollary 4.15 and Corollary 4.16. We conjecture that there is no ST-curve in M_5.

The classification of ST-curves reminds of the conjectures of Coleman and André-Oort that together imply that there should be no Shimura curve *generically* lying in the moduli space of curves M_g for large g. The main theorem can be rephrased saying that there is no non-compact Shimura curve C lying *entirely* in M_g for $g \geq 4$. We emphasize that 'entirely' refers not only to the non-existence of singular fibers with proper Jacobian but also that the Torelli map is unramified if C should happen to contain points in the image of the hyperelliptic locus. After a first version of this paper appeared, it was shown in [ViZu06] that the hypothesis 'non-compact' can be removed. Together these results imply:

Corollary: There is no Shimura curve in A_g for $g \geq 6$ that lies entirely in M_g.

Unfortunately, for Shimura curves lying only generically in the moduli space of curves M_g none of our techniques of flat geometry apply, since such a curve can never be a Teichmüller curve.

Using the characterization in terms of VHS one deduces that a Teichmüller curve can be Shimura only if it arises as square-tiled covering, i.e. with affine group Γ commensurable to $\mathrm{SL}_2(\mathbb{Z})$. The notions are explained in Section 2 and this key step is stated as Corollary 2.3. Given any element of the affine group of a square-tiled covering, one can effectively compute the action of the fundamental group on $H^1(X_0, \mathbb{Z})$ of a fiber X_0 and hence decide whether the covering gives a Shimura curve or not. But since it is hardly clear how the geometry of the covering translates into properties of the action of the affine group, it seems not tractable to identify Shimura curves in this way.

Our proof is thus rather indirect: Being both a Shimura and a Teichmüller curve imposes strong conditions on the geometry of the degenerate fibers. Exploiting these plus some geometric considerations on flat surfaces suffices to treat the case $g \leq 3$. The non-existence for larger g follows from calculating both sides of the Noether formula for the fibered surface $f : \mathcal{X} \to C$ using that we can work in fact on a modular curve $C = X(d)$. We end up with a finite list of possible covering degrees and types of zeros for ω in genus $g = 4$ and $g = 5$. To show that these cases in fact do not exist for $g = 4$, we translate the geometric information about degenerate fibers into conditions on the monodromy representation of the square-tiled coverings.

The Shimura- and Teichmüller curve in M_3 above appears in various guises in the literature: The author heard about its properties independently from Herrlich and Schmitthüsen ([HeSc08]) and Forni ([Fo06]). If we consider it over the modular curve $X(4) \cong \mathbb{P}^1 \smallsetminus \{6 \text{ points}\}$ instead of over $X(2)$ then it admits a stable model. The fibration hence fits in the context of [TaTuZa05], where it is shown that the total space over $X(4)$ is a K3-surface.
The decomposition of the Jacobian of $y^N = x(x-1)(x-t)$ has been studied by many authors, e.g. [Wo88]. Standard criteria for the monodromy groups of hypergeometric differential equation imply that for $N = 4$ and $N = 6$ the family defines indeed a Shimura curve.
In [Gu03] the family for $N = 4$ is studied from the Arakelov viewpoint and [HeSc08] analyze how this Teichmüller curve intersects other Teichmüller curves.

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1. Shimura curves

In this section we give a characterization of Shimura curves by their variation of Hodge structures. The hard part of this characterization, Arakelov equality implies

Shimura curve, is the content of [ViZu04]. These authors were aware of the converse implication, which we need later on, but a proof does not appear in [ViZu04], nor to the author's knowledge anywhere else in the literature. We thus present this proof assuming that the reader is familiar with the basic notions of algebraic groups.

For a reader with background in Teichmüller we suggest to only compare the results of Theorem 1.2 and Theorem 2.2. To make this possible in a self-contained way, we recall some generalities on variations of Hodge structures. Let $f : A \to C$ be a family of abelian varieties over a smooth curve C, let $\overline{C}$ be its smooth compactification and $S := \overline{C} \smallsetminus C$. We denote by $\mathbb{V} = R^1 f_* \mathbb{Z}$ the corresponding local system. Take any completion $f : \overline{A} \to \overline{C}$ with $\overline{A}$ smooth. Then the Deligne extension $\mathbb{V} \otimes \mathcal{O}_C$ to $\overline{C}$ carries a Hodge filtration

$$0 \subset f_* \omega_{\overline{A}/\overline{C}} \subset (\mathbb{V} \otimes \mathcal{O}_C)_{\mathrm{ext}}.$$

The local system $\mathbb{V}$ and the Hodge filtration form a variation of Hodge structures (VHS) of weight one. The graded pieces of the Hodge filtration together with the quotient map induced by Gauss-Manin connection form a Higgs-bundle

$$(\mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}, \Theta : \mathcal{E}^{1,0} \to \mathcal{E}^{0,1} \otimes \Omega_{\overline{C}}(\log S)),$$

where $\mathcal{E}^{1,0} = f_* \omega_{\overline{A}/\overline{C}}$ and $\mathcal{E}^{0,1} = R^1 f_* \mathcal{O}_{\overline{A}}$.

Given such a Higgs bundle, the subbundle with Higgs field $\Theta = 0$ can be split off as a direct summand (see [Ko87]). The remaining part $(\mathcal{F}^{1,0} \oplus \mathcal{F}^{0,1})$ satisfies the Arakelov inequality

$$2 \cdot \deg(\mathcal{F}^{1,0}) \leq \mathrm{rank}(\mathcal{F}^{1,0}) \cdot \deg(\Omega^1_{\overline{C}}(\log(S))).$$

A VHS $\mathbb{L}$ of rank two is called *maximal Higgs* ([ViZu04]), if the corresponding Higgs bundle satisfies Arakelov equality or equivalently, if Θ is an isomorphism.

Definition 1.1. *A* Shimura datum *consists of*

- *i) a reductive algebraic group G defined over $\mathbb{Q}$,*
- *ii) a vector space V with lattice L and a symplectic paring Q, which is integral on L,*
- *iii) a faithful representation $\rho : G \to \mathrm{Sp}(V, Q)$,*
- *iv) a complex structure $\varphi_0 : S^1 = \{z \in \mathbb{C} : |z| = 1\} \to \mathrm{Sp}(V, Q)_{\mathbb{R}}$ such that $Q(x, \varphi_0(i)x) > 0$ for all $x \neq 0$,*

such that

$$\rho(G) \text{ is normalized by } \varphi_0(S^1) \qquad (H_1)$$

Given these data, $K_{\mathbb{R}} = \{g \in G_{\mathbb{R}} | \rho(g)\varphi_0 = \varphi_0\rho(g)\}$ is a maximal compact subgroup of $G_{\mathbb{R}}$. The quotient $\mathcal{X} = G_{\mathbb{R}}/K_{\mathbb{R}}$ is a bounded symmetric domain and $V_{\mathbb{R}}/L$ with complex structure $\rho(g)\varphi_0\rho(g)^{-1}$ defines a holomorphic family of abelian varieties over D.

A Shimura variety *is the inverse system $\{\mathcal{X}/\Gamma\}_{\Gamma}$ where Γ runs over the torsion-free congruence subgroups of $G(\mathbb{Q})$ such that $\rho(\Gamma)$ preserves L.*

The Hodge group $\mathrm{Hg} := \mathrm{Hg}(\varphi_0)$ *is the smallest $\mathbb{Q}$-algebraic subgroup of* $\mathrm{Sp}(V, Q)$ *that contains the image $\varphi_0(S^1)$.*

The Shimura datum (and the Shimura variety) is of Hodge type *if* $G = \mathrm{Hg}(\varphi_0)$.

For a sufficiently small arithmetic subgroup $\widetilde{\Gamma}$ of Sp_{2g} and $\Gamma = \widetilde{\Gamma} \cap \rho(G)$ we have a map

$$j : \mathfrak{X}/\Gamma \ =: C \to A_g^{\widetilde{\Gamma}}$$

into the moduli space of abelian varieties with some level structure $\widetilde{\Gamma}$.

From an embedding j one obviously regains the whole Shimura datum attached to C and A_g and we will henceforth call j a (representative of a) Shimura variety. From now on we exclusively deal with *Shimura curves*, i.e. Shimura varieties of dimension one. In this case the symmetric domain $\mathfrak{X}$ will be the upper half plane $\mathbb{H}$.

Let $f : A \to C$ be the pullback of the universal family over $A_g^{\widetilde{\Gamma}}$ to C. We will always suppose that the monodromies around the cusps of Shimura- and Teichmüller curves are unipotent, replacing Γ by a subgroup of finite index if necessary.

Let X_{Hg} be the $\mathrm{Hg}(\mathbb{R})^+$-conjugacy class in $\mathrm{Hom}_{\mathrm{alg.grp}/\mathbb{R}}(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m, \mathrm{Hg}_{\mathbb{R}})$ containing $\varphi \circ (\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to S^1)$. Here $+$ denotes the topological connected component. In the definition of a Shimura datum we closely followed Mumford ([Mu69], see also [Sa80] for the condition (H_1)). For the reader's convenience we show for comparison that $(\mathrm{Hg}, X_{\mathrm{Hg}})$ is a Shimura datum in the sense of Deligne ([De79, 2.1.1]):
In fact, [De79] Proposition 1.1.14 derives the axioms (2.1.1.1) and (2.1.1.2) from the fact that φ_0 defines a complex structure compatible with the polarization. The axiom (2.1.1.3), i.e. the non-existence of a $\mathbb{Q}$-factor in $\mathrm{Hg}^{\mathrm{ad}}$ onto which h projects trivially, follows from Hg being the smallest $\mathbb{Q}$-subgroup of $\mathrm{Sp}(H^1(F,\mathbb{Q}), Q)$ containing φ_0. By the condition (H_1) the bounded symmetric domains X_{Hg} and $\mathfrak{X}$ coincide.

Theorem 1.2. *The VHS over a Shimura curve decomposes as follows:*

$$R^1 f_* \mathbb{C} =: \mathbb{V}_{\mathbb{C}} = (\mathbb{L} \otimes \mathbb{T}) \oplus \mathbb{U}$$

Here $\mathbb{T}$ and $\mathbb{U}$ are unitary local systems and $\mathbb{L}$ is maximally Higgs, i.e. the corresponding Higgs field is an isomorphism. In particular, the VHS satisfies the Arakelov equality
If C is non-compact then $\mathbb{U}$ splits off over $\mathbb{Q}$ and over an unramified covering of C the unitary local systems become trivial.
Conversely if a family $f : \overline{A} \to \overline{C}$ of abelian varieties satisfies the Arakelov equality, then its VHS decomposes as above and C is a Shimura curve.

Proof. Since Hg is reductive we may split the representation $\mathrm{Hg} \to \mathrm{Sp}(H^1(F,\mathbb{Q}), Q)$ into a direct sum of irreducible representations. We may split off unitary representations over $\mathbb{Q}$ ([Ko87] Proposition 4.11).

Claim: For each of the remaining irreducible representations there is an isogeny $i : \mathrm{SL}_2(\mathbb{R}) \times K \to \mathrm{Hg}_{\mathbb{R}}$, where K is a compact group, such that the composition of i with $\mathrm{Hg}_{\mathbb{R}} \to \mathrm{Sp}(H^1(F,\mathbb{Q}), Q)_{\mathbb{R}}$ is the tensor product of a representation of $\mathrm{SL}_2(\mathbb{R})$ of weight one by a representation of K.

Assuming the Claim, consider a maximal compact subgroup K_1 of $\mathrm{SL}_2(\mathbb{R}) \times K$ that maps to the centralizer of φ_0 under

$$\mathrm{SL}_2(\mathbb{R}) \times K \longrightarrow \mathrm{Sp}(H^1(F,\mathbb{Q}),Q)_{\mathbb{R}}.$$

The double quotient

$$X'_{\mathrm{Hg}} = \Gamma' \backslash (\mathrm{SL}_2(\mathbb{R}) \times K)/K_1$$

is an unramified cover of X_{Hg}. Since a maximal Higgs field is characterized by the Arakelov equality we may as well prove that the pullback variation of Hodge structures has a maximal Higgs field. Since the fundamental group of X'_{Hg} acts via Γ' it follows immediately from the claim and Lemma 2.1 in [ViZu04] that the VHS only consists of unitary and maximal Higgs subsystems.

Proof of the Claim. We first analyze $\mathrm{Hg}^{\mathrm{ad}}$ and the $\mathrm{Hg}^{\mathrm{der}}(\mathbb{R})$-conjugacy class $X^{\mathrm{ad}}_{\mathrm{Hg}}$ of maps $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathrm{Hg}^{\mathrm{ad}}$ containing $(\mathrm{Hg} \to \mathrm{Hg}^{\mathrm{ad}}) \circ \varphi_0$. Note that $X^{\mathrm{ad}}_{\mathrm{Hg}}$ is a connected component of X_{Hg}.
Each $\mathbb{Q}$-factor of $\mathrm{Hg}^{\mathrm{ad}}$ onto which h projects non-trivially contributes to the dimension of X_{Hg}. Since we deal with Shimura curves $\mathrm{Hg}^{\mathrm{ad}}$ is $\mathbb{Q}$-simple by (2.1.1.3). Let

$$\mathrm{Hg}^{\mathrm{ad}}_{\mathbb{R}} = \prod_{i \in I} G_i$$

be its decomposition into simple factors. Then $X^{\mathrm{ad}}_{\mathrm{Hg}} = \prod X_i$ for X_i a $G_i(\mathbb{R})$-conjugacy class of maps $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_i$. For the same reason, only one of the simple factors, say G_1, of $\mathrm{Hg}^{\mathrm{ad}}_{\mathbb{R}} = \prod_{i \in I} G_i$ is non-compact. The possible complexifications $(G_1)_{\mathbb{C}}$ are classified by Dynkin diagrams. The property 'Shimura curve', i.e. dimension one, implies that $G_1 \cong \mathrm{PSl}(2,\mathbb{R})$.

Now we determine the possible representations. The universal cover $\widetilde{G_1} \to G_1$ factors though

$$H := \mathrm{Ker}(\mathrm{Hg} \to \prod_{i \in I \setminus \{1\}} G_i)^0.$$

We apply [De79] Section 1.3 to

$$(G_1, X_{\mathrm{Hg}}) \leftarrow (H, X_{\mathrm{Hg}}) \to (\mathrm{Sp}(H^1(F,\mathbb{Q})), \mathbb{H}_g),$$

where $\mathbb{H}_g$ is the Siegel half space and $g = \dim(F)$. Since a finite-dimensional representation of $\widetilde{G_1}$ factors through $\mathrm{SL}(2,\mathbb{R})$, we conclude that $G \cong \mathrm{SL}(2,\mathbb{R})$. Moreover, such a representation corresponds to a fundamental weight, hence of weight one. Now we let $\widetilde{K}$ be the universal cover of $\prod_{i \in I \setminus \{1\}} G_i$. Since $\mathrm{Hg} \to \mathrm{Hg}^{\mathrm{ad}}$ is an isogeny, there is a lift of the universal cover $\widetilde{K} \to \mathrm{Hg}$. This lift factors through a quotient K of $\widetilde{K}$ such that the natural map $\mathrm{SL}(2,\mathbb{R}) \times K \to \mathrm{Hg}_{\mathbb{R}}$ is an isogeny.

Since we assumed the representation $\mathrm{Hg}_{\mathbb{R}} \to \mathrm{Sp}(H^1(F,\mathbb{Q}),Q)_{\mathbb{R}}$ to be irreducible, also

$$\rho : \mathrm{SL}(2,\mathbb{R}) \times K \to \mathrm{Hg}_{\mathbb{R}} \to \mathrm{Sp}(H^1(F,\mathbb{Q}),Q)_{\mathbb{R}}$$

is irreducible. Let $W \subset H^1(F, \mathbb{R})$ be an irreducible (necessarily weight one) representation of $\mathrm{SL}(2, \mathbb{R}) \times \{\mathrm{id}\}$. Since K is reductive, hence its representations are semisimple, ρ is the tensor product of W and the representation $\mathrm{Hom}_{\mathrm{SL}(2,\mathbb{R})\times\{\mathrm{id}\}}(W, H^1(F, \mathbb{R}))$ of K. This completes the proof of the claim.

Since we have established the Arakelov equality, we may now refer to [ViZu04] to see that the VHS can be grouped together as claimed. The statements in case C non-compact are in [ViZu04], Section 4. □

The equivalence of the above definition of a Shimura curve and the one stated at the beginning of the introduction is due to Moonen ([Mo98] Theorem 4.3).

Remark 1.3. *If the base curve C is not compact, then we may assume that after passing to a finite unramified covering, the unitary system $\mathbb{U}$ is in fact trivial. This is shown in* [ViZu04] *Theorem 0.2.*

2. Teichmüller curves

2.1. **Moduli space of abelian differentials,** $\mathrm{SL}_2(\mathbb{R})$**-action.** Let $\Omega M_g \to M_g$ be the moduli space of abelian differentials, i.e. the vector bundle whose points are *flat surfaces*, i.e. pairs (X, ω) of a Riemann surface X and a holomorphic 1-form on X. The complement of the zero section, denoted by ΩM_g^*, is stratified according to the number and multiplicities of the zeros of ω. This datum $(k_1, \ldots, k_r)$ with $\sum k_i = 2g - 2$ is called the *signature* of ω or of the stratum.

There is a natural action of $\mathrm{SL}_2(\mathbb{R})$ on ΩM_g^* preserving the stratification, defined as follows. Given a complex point (X, ω) and $A \in \mathrm{SL}_2(\mathbb{R})$ we post-compose the local charts of X minus the set of zeros of ω given by integrating ω with the linear map A. The composition maps give new complex charts of a Riemann surface with punctures, that can be removed by Riemann's extension theorem. We thus obtain a new compact Riemann surface Y. The differentials dz on the image of the charts inside $\mathbb{R}^2 \cong \mathbb{C}$ glue to a well-defined holomorphic one-form η on Y. Consequently, we let $A \cdot (X, \omega) = (Y, \eta)$.

2.2. **Teichmüller curves generated by abelian differentials.**

Definition 2.1. *A* Teichmüller curve $C \subset M_g$ *is an algebraic curve in the moduli space of curves, which is the image of the* $\mathrm{SL}_2(\mathbb{R})$*-orbit of a flat surface* (X, ω) *under the projection* $\Omega M_g \to M_g$*. In this case, the flat surface* (X, ω) *is called a* Veech surface.

In this situation we say that C is *generated by* the Veech surface (X, ω). Teichmüller curves are algebraic curves in M_g such that the universal covering map $\mathbb{H} \to T_g$ of the map $C \to M_g$ is a totally geodesic embedding for the Teichmüller metric. In general, one can also define the $\mathrm{SL}_2(\mathbb{R})$-action on the space of quadratic differentials over M_g and include algebraic curves generated by a pair (X, q) with q a quadratic differential on X in the definition of a Teichmüller curve, but we will not do this since we do not deal with these curves in the sequel.

The normalization $C \to M_g$ of Teichmüller curve will still be called Teichmüller curve. This is motivated by the following characterization.

Let $\mathrm{Aff}^+(X,\omega)$ denote the group of orientation preserving diffeomorphisms of X, that are affine with respect to the charts determined by ω. There is a well-defined map $D : \mathrm{Aff}^+(X,\omega) \to \mathrm{SL}_2(\mathbb{R})$, that associates with a diffeomorphism its matrix part. The image is called the *affine group* (or *Veech group*) of (X,ω), usually denoted by Γ. Then the pair (X,ω) generates a Teichmüller curve if and only if Γ is a lattice in $\mathrm{SL}_2(\mathbb{R})$. In this case $C = \mathbb{H}/\Gamma$.

2.3. **Models of the universal family over a Teichmüller curve.** Parallel to the case of Shimura curves we consider the curves $\mathbb{H}/\Gamma_1$ for all $\Gamma_1 \subset \Gamma$ of finite index at the same time. For sufficiently small Γ_1 there is a universal family of curves over $C_1 := \mathbb{H}/\Gamma_1$ and the monodromies around the cusps are unipotent, see the discussion in Section 1 of [Mö06a].

Given a curve C we denote by $\overline{C}$ its closure. For Teichmüller curves C and their unramified coverings defined above will drop the subscripts from now on and -abusing the notations C, $\overline{C}$ etc. - assume that $\overline{f} : \overline{\mathcal{X}} \to \overline{C}$ is a family of stable curves, more precisely the pullback of the universal family.

The total space $\overline{\mathcal{X}}$ is, in general, not a smooth surface. Indeed it has singularities at some of the singular points of the singular fibres of $\overline{f}$. One can resolve these singularities and obtain a smooth surface $\widetilde{\mathcal{X}}$ together with a birational map $\widetilde{\mathcal{X}} \to ol\mathcal{X}$. The price we pay for that is that the induced fibration $\tilde{f} : \widetilde{\mathcal{X}} \to \overline{C}$ has no longer stable but only semistable fibres. See [HM98] Proposition 3.47 and Proposition 3.48 for an algorithm how to compute the stable and semi-stable models and the references in loc. cit. for a general proof.

The advantage of the stable model $\overline{\mathcal{X}}$ is its direct relation to the geometry of Veech surfaces (see Section 2.7), whereas calculations of intersection numbers work without correction terms on $\widetilde{\mathcal{X}}$ only.

2.4. **The characterization of Teichmüller curves by their VHS.** We recall from [Mö06a] a characterization which, together with Theorem 1.2 will strongly restrict the kind of Teichmüller curves that may give rise to a ST-curve.

Theorem 2.2. *If $f : \mathcal{X} \to C$ is the universal family over an unramified covering of a Teichmüller curve, then its VHS looks as follows:*

$$R^1 f_* \mathbb{C} =: \mathbb{V}_{\mathbb{C}} = (\oplus_{i=1}^r \mathbb{L}_i) \oplus \mathbb{M}.$$

Here $\mathbb{L}_i$ are rank-2 subsystems, maximal Higgs for $i = 1$ and non-unitary but not maximal Higgs for $i \neq 1$. In fact $\mathbb{M}$ splits off over $\mathbb{Q}$ and $r = [K : \mathbb{Q}]$, where $K = \mathbb{Q}(\mathrm{tr}(\gamma), \gamma \in \Gamma)$.
Conversely a family of curves $f : \mathcal{X} \to C$, whose VHS contains a rank 2 maximal Higgs subsystem, is the universal family over a (covering of a) Teichmüller curve.

Corollary 2.3. *If $C \to M_g$ is both a Teichmüller curve and a Shimura curve, then $r = 1$.*

Proof. If $r > 1$ then by Theorem 2.2 the VHS over the Teichmüller curve C contains a local subsystem $\mathbb{L}_i$ which is non-unitary but not maximal Higgs. Such a local subsystem does not appear in the decomposition of the VHS over a Shimura curve by Theorem 1.2. □

2.5. **Some Euclidean geometry.** Given a Veech surface (X, ω). The differential ω defines locally a Euclidean coordinate system. With respect to this the slope of a geodesic is well-defined. By abuse of notation we call this slope and all geodesics with this slope a *direction*. Veech dichotomy ([Ve89]) states that each direction that contains a geodesic joining two zeros or one zero to itself (a *saddle connection*) is *periodic* i.e. each geodesic in this direction is periodic or a saddle connection.

A geodesic on a Veech surface X has a *length* with respect to the metric defined by $|\omega|$. The closed geodesics of a periodic direction (say the horizontal one) sweep out cylinders C_i and we denote their core curves by γ_i. The *width* of C_i is defined to be the length of γ_i in the metric $|\omega|$.

2.6. **Square-tiled coverings.** Teichmüller curves with affine group Γ commensurable to $\mathrm{SL}_2(\mathbb{Z})$, i.e. with $r = 1$ arise as unramified coverings of the once-punctured torus by [GuJu00] Theorem 5.9. They are called *square-tiled coverings* or *origamis*. We denote torus covering throughout by $\pi : X \to E$, considering X as a fiber of the family $\mathcal{X}$ and E as a fiber of a family $\mathcal{E}$. Such a covering can be uniquely factored as $\pi = i \circ \pi_{\mathrm{opt}}$, where i is an isogeny and π_{opt} does not factor through non-trivial isogenies. Such a π_{opt} is called *optimal* (sometimes also 'minimal' or 'maximal'). π_{opt} is ramified over torsion points. Given π_{opt} we may choose π such that $i = [n]$ is multiplication by some integer n. The factorization of π will be important when comparing the Néron models of the singular fibers of $\mathcal{X}$ and $\mathcal{E}$ (Proposition 4.14).

For square-tiled surfaces we normalize the length of horizontal and vertical saddle connections to be integer-valued by demanding that the horizontal and vertical closed geodesics in $E \cong \mathbb{C}/\mathbb{Z}[i]$ have length one.

2.7. **Degeneration.** The following proposition shows that many boundary components of the moduli space of stable curves $\overline{M_g}$ are not hit by the the closure of a Teichmüller curve in $\overline{M_g}$.

By [Ve89] (see also [MaTa02]) the only degenerate fibers of Teichmüller curves, the stable curves over the cusps $\overline{C} \smallsetminus C$ are obtained as follows. Rotate (X, ω) such that the horizontal direction decomposes into cylinders C_i. Then flow along the Teichmüller geodesic flow by applying $\mathrm{diag}(e^t, e^{-t})$ to (X, ω) for $t \to \infty$. By [Ma74] the stable model of the fiber $\mathcal{X}_c$ of f is obtained by 'squeezing the core curves γ_i' of the C_i to points. More precisely, along this ray the cylinders acquire larger and larger moduli. The limit stable curve $\mathcal{X}_c$ is obtained by glueing in place of each cylinder C_i two half-infinite cylinders. In particular, topologically the irreducible components of $\mathcal{X}_c$ are obtained cutting along the γ_i.

Proposition 2.4. *The universal family over a Teichmüller curve $f : \mathcal{X} \to C$ does not contain singular fibers whose Jacobian is proper, i.e. an abelian variety without toric part.*

Proof. We give two proofs: The VHS depends only on the family of Jacobians. If the Jacobian is proper for some $s \in S := \overline{C} \setminus C$ the Higgs field $\Theta : E^{1,0} \to E^{0,1} \otimes \Omega^1_{\overline{C}}(\log S)$ factors through $E^{0,1} \otimes \Omega^1_{\overline{C}}(\log(S - s))$. This contradicts that $\mathbb{L}_1$ is maximal Higgs.
Or by topology: The Jacobian of a stable curve is proper if and only all the nodes are separating. As described in Section 2.5 the nodes of the stable curves in the family over a Teichmüller curves are obtained by squeezing the core curves of all cylinders in a fixed direction. But a core curve of a cylinder can never be separating. □

2.8. **Sections of the family of curves over a Teichmüller curve.** For a square-tiled covering the preimages of torsion points are *periodic*, i.e. they have a finite $\mathrm{Aff}^+(X, \omega)$-orbit ([GuJu00], [Mö06b]). If we let Γ_2 be the the image under the derivative D of the stabilizer of a point in this orbit, then Γ_2 is of finite index in Γ and the torsion point defines a section of the pullback of the family f over $\mathbb{H}/\Gamma_1$ to $\mathbb{H}/(\Gamma_1 \cap \Gamma_2)$. We may do so for finitely many torsion points at the same time. We still want to call the base curve C for simplicity of notation and just recall at appropriate places that we passed to a finite index subgroup of Γ.

In particular, the zeros Z_i of ω are periodic points on the Veech surface. Let S_i denote these sections.

Lemma 2.5. *The sections S_i intersect the singular fibers of the family $\overline{f}$ in smooth points of the singular fiber.*

Consequently, we may speak of the zero Z_i on a singular fiber $\mathcal{X}_c$ of $\overline{f}$, meaning the intersection of S_i with $\mathcal{X}_c$.

Proof. This follows from the description of the degeneration to any singular fiber given in Section 2.7. In fact, the zeros of ω cannot lie in the interior of cylinders but on their boundary. Consequently, a neighborhood of the zeros of ω remains untouched when replacing the finite cylinders with half-infinite cylinders. (The singular points of the singular fibers correspond to the infinitely far points of the half-infinite cylinders.) □

In fact, the same argument show the follows more general fact. Consider the factorization of the square-tiled covering as $\pi = \imath \circ \pi_{\mathrm{opt}}$. Let $B \subset E$ the set of points over which π_{opt} is branched. Any preimage $P \in \pi_{\mathrm{opt}}^{-1}(B)$ is a torsion point on X and thus defines as section R of f, again after passing to a finite covering. This section R hits singular fibers in smooth points, as do the S_i.

3. Cyclic coverings

Families of cyclic coverings are easily understandable families of curves with interesting variation of Hodge structures. They will turn out to produce a number of Shimura and ST-curves. As we will see in the next section, these are all examples with possible exceptions in $g = 5$.

Fix integers a_i and N with $a_i \not\equiv 0 \mod N$, $a_1 + a_2 + a_3 \not\equiv 0 \mod N$. For a family of cyclic coverings

$$\mathcal{Y}_{N,\underline{a}} : y^N = x^{a_1}(x-1)^{a_2}(x-t)^{a_3}$$

the VHS decomposes into eigenspaces $\mathbb{L}_i$ where the automorphism

$$\varphi : (x,y) \mapsto (x, \zeta_N y)$$

acts by the i-th power of ζ_N.

Their dimension and signature can be determined combinatorially from N and the 'type' (a_1, a_2, a_3). This is known since the work of Chevalley-Weil and is recalled for 4-point covers e.g. in Lemma 3.1 of [BM05]. The family $\mathcal{Y}_{N,(a_1,a_2,a_3)}$ defines a Shimura curve, if and only if on all the decomposition pieces the monodromy action is by a Fuchsian group or by a unitary group. Such a curve is moreover a Teichmüller curve, if it contains no singular fiber with proper Jacobian.

For the special case $a_i = 1$ for all i we obtain the following list of ST-curves.

Theorem 3.1. *The family of curves $\mathcal{Y}_{N,(1,1,1)}$ defines a ST-curve if and only if $N \in \{2, 4, 6\}$.*

Proof. Using the theory of admissible coverings one quickly shows that the degenerate fibers for $t = 0$ consists of two curves isomorphic to $y^N = x(x-1)$ and $y^N = x(x-1)^{N-3}$ respectively, intersecting each other at $\gcd(N-2, N)$ normal crossings. Since degenerate fibers of the universal family over a Teichmüller curve do not have separating nodes, this implies for ST-curves that N is even.

Since $a_1 = a_2 = a_3 = 1$, [BM05] Lemma 3.1 implies that for $i \leq N/3$ and for $i \geq 2N/3$ the Hodge filtration on the local system $\mathbb{L}_i$ is trivial, i.e. these local systems are unitary. Since the local system $\mathbb{L}_{N/2}$ pulls back from the family $y^2 = x(x-1)(x-t)$ of elliptic curves, it is maximal Higgs (with respect to the set $\{0, 1, \infty\}$ in the language of [BM05]). This already implies that for $N = 4$ and $N = 6$ the family is a Shimura curve. Since the set of singular fibers of the family $\mathcal{Y}_{N,(1,1,1)}$ is exactly $\{0, 1, \infty\}$, this family is a Teichmüller curve.

It remains to show that for even $N > 6$ one of the $\mathbb{L}_i$ is neither maximal Higgs nor unitary. For this purpose we take $i = N/2 + 1$. There is a general criterion for checking if the monodromy group of a hypergeometric differential equation (not only of second order) is unitary. Hypergeometric differential equations of order n are determined by a standard set of parameters $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, see e.g. the introduction of [BH89]. These α_i and β_i are easily calculated from the local parameters (compare the proof of [BM05] Lemma 3.1) of the differential equation. These local parameters are, for the chosen i, given by $\{0, 2/N\}$ at $x = 0$, $\{0, 2/N\}$ at $x = 1$ and $\{(N-6)/2N, (N-2)/2N\}$. We obtain, since $N > 6$,

$$0 < \alpha_1 = \frac{N-6}{2N} < \alpha_2 = \frac{N-2}{2N} < \beta_1 = \frac{2N-4}{2N} < \beta_2 = 1.$$

The criterion of [BH89] Corollary 4.7 is that $\exp(2\pi I(\cdot))$ applied to the sets $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ interlace on the unit circle. The above ordering shows that this is not the case.

In order to show that $\mathbb{L}_{N/2+1}$ is not maximal Higgs we use the uniqueness of a maximal Higgs sub-variation of Hodge structures of a family of curves, see [Mö06a] Remark 2.5.

□

For the last argument one can, alternatively, calculate directly the Lyapunov exponents (see Section 5) of $\mathbb{L}_{N/2+1}$ using [BM05] Proposition 3.4.

For $N = 4$ we next show that $\mathcal{Y}_{4,(1,1,1)}$ defines a Teichmüller curve by exhibiting the corresponding square-tiled covering. The same thing is easily done for $N = 6$. The fact that, at least for $N \geq 10$ the families $\mathcal{Y}_{N,(1,1,1)}$ do not define ST-curves follows of course also from the main theorem, which to be proven in the next section.

For $N = 5$ and $N = 7$ the families $\mathcal{Y}_{N,(1,1,1)}$ give Shimura curves in the closure of M_g. This was first observed in [dJN91]. One can also use the methods of [BM05] Section 3 to show that the $\mathbb{L}_i$ are either unitary or maximal Higgs. Again, for odd $N > 7$, there exists a subsystem $\mathbb{L}_i$ that is neither maximal Higgs nor unitary.

3.1. **The ST-curve in genus 3.** We quickly present a ST-curve in genus 3, as we will see the only one, as a square-tiled covering. This way we can tell in the next section that a remaining configuration is indeed this ST-curve. More details on this curve are in [HeSc08].

Proposition 3.2. *The family of curves $y^4 = x(x-1)(x-t)$ corresponds to the square-tiled surface shown in Figure 1. The subgroup $\pi_1(E^*)$ defining the covering is characteristic and has as quotient group the Quaternion group Q of order 8. The Veech group of the square-tiled surface is $\mathrm{SL}_2(\mathbb{Z})$.*

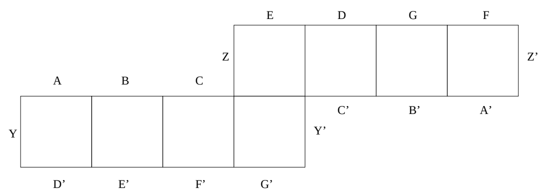

FIGURE 1. The Teichmüller- and Shimura curve in genus 3

Proof: Denote the covering map corresponding to the above figure by $\pi : X \to E$. One checks that the covering group is indeed Q. In addition to that the elliptic involution lifts to an involution φ of X. This can be checked graphically since 180° rotation of each square gives a well-defined automorphism of X. In particular, the hyperelliptic involution does not belong to the covering group Q and the automorphism group of X has at least order 16. Counting fixed points one checks that $X/ < \varphi >$ is of genus one. If X was hyperelliptic, the involution would have to be the lift of the elliptic involution of E. Hence the generic fiber over the above Teichmüller curve is not hyperelliptic.

By [KuKo79] there are only two families of curves of genus three with an automorphism group of order 16. The one with hyperelliptic generic fiber can be ruled out and it remains the above family.

To determine the Veech group, remark that characteristic subgroups of $\pi_1(E^*)$ always have Veech group $\mathrm{SL}_2(\mathbb{Z})$ (see e.g. [He06]). $\square$

4. Both Shimura- and Teichmüller curve

The proof of the following classification result will occupy the whole section:

Theorem 4.1. *For $g = 2$ and $g \geq 6$ there is no Shimura curve that is also a Teichmüller curve.*
In both M_3 and in M_4 there is only one curve which is both a Shimura- and a Teichmüller curve. Its universal family is given by

$$y^4 = x(x-1)(x-t)$$

in M_3 and respectively in M_4 by

$$y^6 = x(x-1)(x-t).$$

In both cases $t \in \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$.

We recall that we abbreviate Shimura curves that are also Teichmüller curves as *ST-curves.* We now sketch the main steps of the proof of this theorem. Existence statements were shown in the preceding Section 3, so it remains to show the non-existence statements.

Strategy of the proof of Theorem 4.1: From Corollary 2.3 and Section 2.6 it follows that a ST-curve is necessarily a square-tiled covering, i.e. that $r = 1$ or equivalently $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. Moreover, by Theorem 1.2 the local system $\mathbb{M}$ appearing in Theorem 2.2 has to be unitary. Since the ST-curve is not compact and by Remark 1.3, there is an abelian subvariety A of dimension $g-1$ such that $A \times C \hookrightarrow \mathrm{Jac}\, \widetilde{\mathcal{X}}/\overline{C}$. The abelian variety A is called *fixed part* of the fibration. Fibered surfaces with a fixed part of dimension $g-1$ have been called *maximally irregular* in [Mö05a].

In Section 4.1 we show, for any $g \geq 3$, that all the singular fibers of the stable model $\overline{\mathcal{X}}$ are of a very particular type. Using the dictionary between the cylinder decomposition of a Veech surface generating a ST-curve and the fibers over the cusps of the ST-curve, we prove in Section 4.2 that for $g = 3$ the ST-curve generated by the flat surface in Figure 1 is the only ST-curve. The proof has the advantage to use flat geometry only, but Section 4.2 is strictly speaking not necessary for a complete proof.

In Section 4.3 we exploit the Noether formula (or equivalently Riemann-Roch for $f : \mathcal{X} \to C$). This formula involves the self-intersection of the relative dualizing sheaf of f. We are able to calculate this self-intersection (Lemma 4.12 and Proposition 4.13) exploiting two facts. By Section 4.1 the family f has no 'bad' singular fibers (in the sense that the map in the proof of Lemma 4.12 would acquire a cokernel supported on part of the singular fiber) and moreover we know the self-intersections of sections of f (Lemma 4.11) since C arises as square-tiled surface. The positivity of the relative dualizing sheaf now implies the non-existence for $g \geq 6$.

To obtain even more precise results for $g = 4$ and $g = 5$ we have to relate the contributions of the singular fibers (the $\Delta\chi_{\mathrm{top}}(F)$ in (1)) – that we previously just estimated – to the degree d_{opt} of the covering π_{opt} as defined in Section 2.6.

To prove the classification for $g = 4$ we use that by [Mö05a] the group $\mathrm{SL}(X, \omega)$ is contained the congruence subgroup $\Gamma(d)$ and that we may work over $\mathbb{H}/\Gamma(d)$ all along. The geometric reformulation of this fact is given in Corollary 4.16: We may assume that X is a branched cover of a square such that all cylinders in all directions have the same widths.

On the other hand, from the Noether formula we obtain a quite restrictive list of the singularities of a Veech surface generating a ST-curve for $g = 4$ or $g = 5$. The list of singularities of course directly translates into the branching behavior of $\pi : X \to E$, where X is a fibre of $\mathcal{X}$ and where π is the covering that makes X square-tiled. The remainder is a combinatorial discussion whether such a covering with the cylinder condition from Corollary 4.16 exists. $\square$

4.1. **Degeneration of ST-curves.** We work no longer in $g = 3$ but we will only suppose $g > 1$ here.

Lemma 4.2. *The singular fibers of the smooth model of the universal family over a ST-curve $\tilde{f}$ have geometric genus $g - 1$, hence n components and n nodes for some n. If we consider the stable model $\overline{f}$ of the universal family, n is at most $g - 1$.*

Proof. The Jacobian of the singular fiber is a semiabelian variety with non-trivial toric part by Proposition 2.4. The abelian part has dimension equal to the geometric genus of the singular fiber. Since the fixed part A of dimension $g - 1$ injects into the Jacobian of each fiber, the geometric genus of the singular fiber equals $g - 1$. Hence the dimension of the toric part is one and this dimension equals the rank of the fundamental group of the dual intersection graph of the singular fiber. A graph with fundamental group of rank one and without separating nodes (compare the topological argument in the proof of Proposition 2.4) is a ring. $\square$

We now look at the cylinder decomposition of a fixed Veech surface (X, ω) generating a ST-curve in a fixed periodic direction, say the horizontal one. As in Section 2.5 let C_i (for $i = 1, \ldots, m$) denote the cylinders in this direction.

Lemma 4.3. *For an appropriate numbering of the cylinders, the saddle connections bounding C_i from the above will bound C_{i+1} from below (subscripts taken mod n). Consequently, all cylinders of (X, ω) in a fixed direction have the same width.*

Proof. By the previous lemma the number of components of the singular fiber equals the number of nodes, which equals (again by the description in [Ma74]) the number of cylinders. By Proposition 2.4 this is only possible if the first assertion is true and the second assertion is a consequence of the first. $\square$

In the sequel we use that we can retrieve singularities of ω also on the singular fibers, see Section 2.8.

Lemma 4.4. *If a component of a singular fiber of $\overline{f}$ contains zeros Z_i of order k_i for $i = 1, \ldots, \tilde{s}$, the geometric genus is $\sum_{i=1}^{\tilde{s}} k_i/2$. In particular the geometric genus of a component of the stable model $\overline{f}$ is always positive and $\sum_{i=1}^{\tilde{s}} k_i$ is always even.*

Proof. Apply Gauß-Bonnet to a surface with 2 boundary components. □

Lemma 4.5. *Each direction on a Veech surface generating a ST-curve contains at least two cylinders.*

Proof. If not, we may find a transverse direction – without loss of generality the vertical one – with a cylinder of width one. By Lemma 4.3 all cylinders in this direction have this property. This means that the square-tiled covering is abelian and $g(X) = 1$. □

Lemma 4.6. *A Veech surface generating a ST-curve has no saddle connection from a simple zero Z to itself.*

Proof. Suppose that such a saddle connection S connection exists, say on the top boundary of a horizontal cylinder C_1 and on the lower boundary of C_2. Consider the saddle connection S' on the left of S, viewed from C_1. By Lemma 4.3 there must also be a saddle connection on C_2, whose right end is Z. But there are only two saddle connections with this property: S and the saddle connection S'' on the left of S, viewed from C_2. But if we have $S' = S''$ then Z is not be a zero at all. □

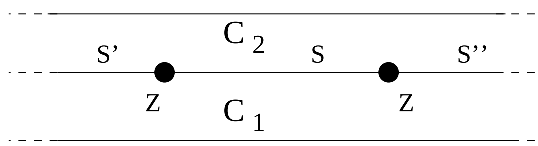

FIGURE 2. Saddle connection S joining a simple zero Z to itself

Remark 4.7. There are no ST-curves in genus two. This will follow from Corollary 4.15, but there are simpler proofs. E.g. [Xi85] Theorem 3.16 implies that a genus two fibration with one-dimensional fixed part has always singular fibers with smooth Jacobian, a contradiction to Proposition 2.4.

4.2. **Genus three.**

We suppose that $g = 3$ and show that the square-tiled covering of Figure 1 generates the only ST-curve in M_3. We give here a simple geometric proof not relying on the intersection numbers on $\widetilde{\mathfrak{X}}$ as in Section 4.3.

Let (X, ω) be a Veech surface generating a ST-curve in M_3. By Lemma 4.5 and Lemma 4.2 each periodic direction on (X, ω) has precisely two cylinders, and, by Lemma 4.2 and Section 2.7, the corresponding singular fiber of the stable model has precisely two components.

Lemma 4.8. *All zeros of ω are simple.*

Proof. Suppose that there is a zero Z of order three or four and choose any periodic direction. The corresponding singular fiber of the stable model has a component with no zeros or with only a simple zero. This contradicts Lemma 4.4.

Suppose there is a zero Z of order two and choose a saddle connection that connects Z to some other zero Z'. Again the corresponding singular fiber of the stable model has a component containing Z and Z', while the other component has at most one simple zero. This contradicts again Lemma 4.4. □

Lemma 4.9. *In each parabolic direction all saddle connections have the same length and the cylinders have the same height.*

Proof. Suppose we fix a parabolic direction where this is not the case. Let A be one of the longest saddle connections in this direction. In particular $\ell(A) > w/4$, where ℓ denotes the length of the saddle connection, normalized as in Section 2, and w the width of any of the cylinders in this direction. By Lemma 4.8 the square-tiled covering looks roughly as follows (see Figure 3):

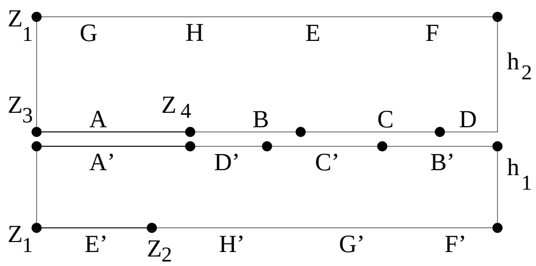

FIGURE 3. A potential ST-curve in genus 3. The location of the remaining zeros on the top and bottom line is not indicated since any location leads to contradiction.

The gluing A to A' etc. is imposed by the fact that we have simple zeros and by Lemma 4.6. The top line consists of four saddle connections E, F, G and H, that are glued to E', H', G', F' in this cyclic order on the bottom line. We have not drawn the square tiling, but we may assume that the lengths of A, B, E, F, etc., h_1 and h_2 are integers. Turning the figure upside down we may suppose $h_1 \leq h_2$.

We consider straight lines emanating from Z_1 passing through A and check which part on the upper boundary of the upper cylinder is hit. By Lemma 4.6 the two occurrences of Z_1 on the top line have to be in the segment not hit by such lines. This segment is of length $w - \frac{h_1+h_2}{h_1}\ell(A)$. It contains say E and F. Hence $\ell(E)+\ell(F)$ equals at most $w - \frac{h_1+h_2}{h_1}\ell(A)$. We may suppose that

$$\ell(E) \leq \frac{1}{2}(w - \frac{h_1 + h_2}{h_1}\ell(A)) < \ell(A),$$

where the strict inequality comes from the assumption that we want to contradict. We suppose moreover that the 'short' saddle connection E' is the one drawn in the Figure 3, switching the roles of Z_1 and Z_2 if necessary.

Now consider the lines emanating from Z_2 passing through A. The segment on the top line hit by these lines does not contain Z_2. The intersection I with the corresponding 'hit segment' from Z_1 does not contain any zero. One calculates

$$\ell(I) = \frac{h_1 + h_2}{h_1}\ell(A) - \frac{h_2}{h_1}\ell(E) = \ell(A) + \frac{h_2}{h_1}(\ell(A) - \ell(E)) > \ell(A).$$

This contradicts the maximality of A. $\square$

Proof of Theorem 4.1 for $g = 3$: By the Lemmas 4.3 and 4.9 we know that the square-tiled covering is of degree 8, maybe post-composed by unnecessary isogenies. Since all zeros are simple, since there is no saddle connection joining a zero to itself and since the, say, horizontal direction has two cylinders of same length, the horizontal direction consists of two cylinders of width 4. There is only one way to glue these cylinders to obtain simple singularities, the way shown in Figure 1. $\square$

4.3. Consequences of the Noether-formula on $\widetilde{\mathfrak{X}}$.

We first show that there is an unramified covering of a ST-curve which is a modular curve, i.e. the quotient of the upper half plane by a congruence subgroup.

Lemma 4.10. *A ST-curve C admits an unramified covering which is the modular curve $X(d)$, where $X(d) = \mathbb{H}/\Gamma(d)$. Moreover we can arrange that $d = \deg(\pi)$.*

Proof. By Theorem 1.6 of [Mö05a] a universal family of principally polarized abelian varieties of dimension g having as fixed part some abelian variety A of dimension $g - 1$ exists over some $X(d_1)$. (In fact, $d_1 = d_{\mathrm{opt}}$, be we will not need this.) Hence there is some covering of the Teichmüller curve C such that the moduli map $C \to A_g$ factors through $X(d_1)$.

What remains to show is that in case the family comes from a ST-curve (i.e. consists exclusively of Jacobians) the family of curves exists over some $X(d)$, i.e. $C \to M_g$ factors through $X(d)$.

By loc. cit. Proposition 1.7 the generic fiber of a ST-curve with $g \geq 3$ cannot be hyperelliptic. The action of $\mathrm{SL}_2(\mathbb{R})$ on the moduli space of one-forms preserves the property of being hyperelliptic. Consequently, all the smooth fibres of the family over the ST-curve are not hyperelliptic. (But some of the singular fibres might be in the closure of the hyperelliptic locus – and this will be indeed the case.)

We claim that $C \to A_g$ factors through M_g, if we take $\overline{C} \to \overline{X(d_1)}$ ramified of order two at all the cusps. Indeed, the failure of infinitesimal Torelli (see [OS80] Corollary 2.8 and the explanation on p. 168) at the hyperelliptic locus in $g \geq 3$ might force us to pass to a ramified cover, but a double cover is always sufficient. Since the smooth fibres are not hyperelliptic such a double covering is unramified over the (part in the interior of moduli space of the) ST-curve. For large enough d/d_1 the covering $\overline{X(d)} \to \overline{X(d_1)}$ is ramified at all cusps of $\overline{X(d_1)}$ of even order. Hence the ST-curve exists over $X(d)$ for some d.

The second statement can be arranged by post-composing π with an isogeny. $\square$

In the sequel we will sometimes enlarge d further if necessary: We suppose that d is large enough such that the base $X(d)$ has positive genus. Since the fibers of $f : \mathcal{X} \to C$ have genus greater than two, $\mathcal{X}$ has a unique minimal model and this hypothesis on $X(d)$ ensures that the smooth relatively minimal model for $f : \mathcal{X} \to C$ is in fact minimal.

On the smooth minimal model $\tilde{f} : \widetilde{\mathcal{X}} \to \overline{C}$ we have the Noether equality

$$12\chi(\mathcal{O}_{\widetilde{\mathcal{X}}}) - c_2(\omega_{\widetilde{\mathcal{X}}}) = c_1(\omega_{\widetilde{\mathcal{X}}})^2.$$

For a fibered surface with fiber genus g and base genus b we have by Riemann-Roch and the Leray spectral sequence (see e.g. [Xi85])

$$\begin{array}{rcl} \chi(\mathcal{O}_{\widetilde{\mathcal{X}}}) & = & \deg \tilde{f}_*\omega_{\widetilde{\mathcal{X}}/\overline{C}} + (g-1)(b-1) \\ c_2(\omega_{\widetilde{\mathcal{X}}}) & = & \sum_{F \text{ sing.}} \Delta\chi_{\mathrm{top}}(F) + 4(g-1)(b-1) \\ c_1(\omega_{\widetilde{\mathcal{X}}})^2 & = & \omega^2_{\widetilde{\mathcal{X}}/\overline{C}} + 8(g-1)(b-1). \end{array}$$

We will hence use the equality in the form

$$12 \deg \tilde{f}_*\omega_{\widetilde{\mathcal{X}}/\overline{C}} - \sum_{F \text{ sing.}} \Delta\chi_{\mathrm{top}}(F) = \omega^2_{\widetilde{\mathcal{X}}/\overline{C}}. \qquad (1)$$

Here the sum runs over all singular fibers F of $\tilde{f}$ and $\Delta\chi_{\mathrm{top}}$ denotes the difference of the topological Euler characteristic of F and $2-2g$.

It is well-known (e.g. [Sh71]) that

$$g(X(d)) = (d-6)\Delta_d + 1, \text{ where } \Delta_d = \begin{cases} \frac{d^2}{24}\prod_{p|d}(1-\frac{1}{p^2}) & d \geq 3 \\ \frac{1}{4} & d = 2 \end{cases}$$

and that the number of cusps is

$$|S| := |\overline{X(d)} \smallsetminus X(d)| = 12\Delta_d, \quad \text{hence} \quad \deg \Omega^1_{\overline{C}}(\log S) = 2d\Delta_d.$$

For a Shimura curve with fixed part of dimension $g-1$ the Arakelov equality states that

$$\deg \tilde{f}_*\omega_{\widetilde{\mathcal{X}}/\overline{C}} = \frac{1}{2}\deg \Omega^1_{\overline{C}}(\log S) = d\Delta_d.$$

We start calculating $c_1(\omega_{\widetilde{\mathcal{X}}})^2$ and $c_2(\omega_{\widetilde{\mathcal{X}}})$ for a square-tiled covering. Note that the map $\varphi : \widetilde{\mathcal{X}} \to \overline{\mathcal{X}}$ from the smooth semistable model to the stable model only contracts (-2)-curves (i.e. twice punctured spheres in the fibers of $\tilde{f}$). Hence $\omega_{\widetilde{\mathcal{X}}/\overline{C}} = \varphi^*\omega_{\overline{X}/\overline{C}}$. Let $h : \mathcal{E} \to X(d) = C$ be the universal family of elliptic curves with full level-d-structure. We extend this family to a smooth minimal model $\tilde{h} : \widetilde{\mathcal{E}} \to \overline{C}$.

Recall from Section 2.8 the the definition of the sections S_i and R. We need to calculate the self-intersection numbers of these sections. In order to do so we have to perform some base changes and blowups, that will not appear in the formulae at the end of the day.

The sections S_i and R are defined over some unramified cover $\psi : C_{\mathrm{sec}} \to X(d)$, not necessarily given by a congruence subgroup. Let

$$\psi_X : \widetilde{\mathcal{X}} \times_{\overline{C}} \overline{C}_{\mathrm{sec}} \to \widetilde{\mathcal{X}}$$

be the covering of the surfaces induced by the base change ψ. There is a blowup $b : \mathcal{Y} \to \widetilde{\mathcal{X}} \times_{\overline{C}} \overline{C}_{\mathrm{sec}}$ in the singularities of the singular fibers of $\widetilde{\mathcal{X}} \times_{\overline{C}} \overline{C}_{\mathrm{sec}} \to \overline{C}_{\mathrm{sec}}$ such that $\mathcal{Y}$ is smooth and such that the square-tiled covering map π extends to a map

$$\pi_{\mathrm{sec}} : \mathcal{Y} \to \widetilde{\mathcal{E}_{\mathrm{sec}}} := \widetilde{\mathcal{E}} \times_{\overline{C}} \overline{C}_{\mathrm{sec}}.$$

By definition of $\mathcal{Y}$ there is a blow down $b_2 : Y \to \widetilde{\mathcal{X}_{\mathrm{sec}}}$ to the minimal semistable model $\widetilde{\mathcal{X}_{\mathrm{sec}}}$ over $\overline{C}_{\mathrm{sec}}$, through which b factors.

Suppose that the Teichmüller curve lies in the stratum with signature $(k_1, \ldots, k_s)$, i.e. in each smooth fiber F of f the zeros of $\omega|_F$ have multiplicity $k_1, \ldots, k_s$.

Lemma 4.11. *The sections S_i on $\widetilde{X_{\mathrm{sec}}}$ have self-intersection number*

$$S_i^2 = \frac{-\deg \tilde{f}_* \omega_{\widetilde{X_{\mathrm{sec}}}/\overline{C_{\mathrm{sec}}}}}{k_i + 1}.$$

Proof. Kodaira has calculated ([Kd63] (12.6) and (12.7)) that the zero-section σ_0 of $\mathcal{E}$ has self-intersection

$$\sigma_0^2 = -d\Delta_d = -\deg \tilde{f}_* \omega_{\widetilde{\mathcal{X}}/\overline{C}},$$

hence on the pullback surface $\widetilde{\mathcal{E}_{\mathrm{sec}}} := \widetilde{\mathcal{E}} \times_{\overline{C}} \overline{C_{\mathrm{sec}}} \to \overline{C_{\mathrm{sec}}}$ we have

$$(g_{\mathcal{E}}^* \sigma_0)^2 = -\deg \tilde{f}_* \omega_{\widetilde{\mathcal{X}_{\mathrm{sec}}}/\overline{C_{\mathrm{sec}}}},$$

where $g_{\mathcal{E}} : \widetilde{\mathcal{E}_{\mathrm{sec}}} \to \widetilde{\mathcal{E}}$. The blowup b_2 is an isomorphism outside the singularities of the singular fibers of f and S_i does not hit them, see the description of the degeneration in Section 2. Hence

$$S_i^2 = (b_2^* S_i)^2 = \frac{1}{k_i + 1} (b_2^* S_i \cdot \pi^* g_{\mathcal{E}}^* \sigma_0) = \frac{-\deg f_* \omega_{\widetilde{X_{\mathrm{sec}}}/\overline{C}_{\mathrm{sec}}}}{(k_i + 1)}.$$

□

The same argument applies to the section induced by an unramified torsion point R and yields $R^2 = -d\Delta_d$. In the sequel we denote the families of curves over $\overline{C}_{\mathrm{sec}}$ still by $\overline{f}$ or $\tilde{f}$, without the subscript sec.

Lemma 4.12. *On the smooth minimal model $\tilde{f} : \widetilde{\mathcal{X}}_{\mathrm{sec}} \to \overline{C}_{\mathrm{sec}}$ of a ST-curve coming from a square-tiled covering we have*

$$\omega_{\widetilde{\mathcal{X}}_{\mathrm{sec}}/\overline{C}_{\mathrm{sec}}} = \mathcal{O}_{\widetilde{\mathcal{X}}_{\mathrm{sec}}} \left(\sum_{i=1}^{s} k_i S_i + \deg \tilde{f}_* \omega_{\widetilde{\mathcal{X}_{\mathrm{sec}}}/\overline{C}_{\mathrm{sec}}} \cdot F \right),$$

where F denotes a fiber.

Proof. The generating differential ω^0 defines a subbundle $\mathcal{L} \subset \tilde{f}_* \omega_{\widetilde{\mathcal{X}}_{\mathrm{sec}}/\overline{C}_{\mathrm{sec}}}$. By definition of the k_i and S_i, the cokernel of $\tilde{f}^* \mathcal{L}(\sum k_i S_i) \to \omega_{\widetilde{\mathcal{X}}_{\mathrm{sec}}/\overline{C}_{\mathrm{sec}}}$ is supported in the singular fibers.

We claim that the cokernel does not contain components of singular fibers but only the complete fibers. Once the claim is established the number of these fibers is $-R^2 = \deg \tilde{f}_* \omega_{\widetilde{X_{\rm sec}}/\overline{C}_{\rm sec}}$ by the adjunction formula.

To prove the claim, suppose the cokernel contains a connected divisor B with multiplicity $\mu > 0$, contained in a singular fiber but not equal to a singular fiber. Lemma 4.4 also applies to B, since the dual graph of B contains no loops and yields $g(B) = \sum_{i \in I_B} k_i/2$, where the sum is over the sections S_i, $i \in I_B$ that hit the component B. On the other hand $B^2 = -2$, since B hits the rest of the fiber in two points by Lemma 4.2. Now the adjunction formula says

$$B \cdot \omega_{\widetilde{\mathfrak{X}}_{\rm sec}/\overline{C}_{\rm sec}} = 2g(B) - 2 - B^2 = 2g(B) = \sum_{i \in I_B} k_i.$$

On the other hand, by our hypothesis on the cokernel

$$B \cdot \omega_{\widetilde{\mathfrak{X}}_{\rm sec}/\overline{C}_{\rm sec}} = \mu B^2 + \sum_{i \in I_B} k_i$$

This is a contradiction. □

Although we will calculate more precisely, let us note here an immediate corollary

Proposition 4.13. *ST-curves exist only for $g \leq 6$.*

Proof. Since $g \geq 2$ and since we may suppose that $g(C_{\rm sec}) \geq 2$ by enlarging d, the surface $\widetilde{\mathfrak{X}}$ is of general type and hence $\omega_{\widetilde{\mathfrak{X}}_{\rm sec}/\overline{C}_{\rm sec}}$ is nef (Arakelov's theorem, see e.g. [Be82]). Since there is at least one singular fiber we have, using (1),

$$\begin{array}{rcl} 12 d \Delta_d & > & (\omega_{\widetilde{\mathfrak{X}}_{\rm sec}/\overline{C}_{\rm sec}} \cdot d\Delta_d F) + (\omega_{\widetilde{\mathfrak{X}}_{\rm sec}/\overline{C}_{\rm sec}} \cdot (\sum_{i=1}^s k_i S_i)) \\ & \geq & (\omega_{\widetilde{\mathfrak{X}}_{\rm sec}/\overline{C}_{\rm sec}} \cdot d\Delta_d F) = (2g-2) d\Delta_d. \end{array}$$

Solving for g gives the desired inequality. □

We now come to the calculation of $\Delta\chi_{\rm top}$. For this purpose, it will be more convenient to work with family of curves over $C = X(d)$ instead of the covering over $C_{\rm sec} \to C$. Recall that from Section 2.6 that we may suppose $\pi = [n] \circ \pi_{\rm opt}$. With this normalization, the image of $\pi_{\rm opt}$ is the universal family of elliptic curves over $X(d)$, not just any family isogenous to that one.

Proposition 4.14. *For the smooth minimal model $\widetilde{\mathfrak{X}}$ of a ST-curve we have for each of the $12\Delta_d$ singular fibers*

$$\Delta\chi_{\rm top} = d/d_{\rm opt}.$$

Proof. By Lemma 4.2 the singular fibers of $\tilde{f}$ consist of a ring of say $\tilde{m}$ smooth curves. In that case $\Delta\chi_{\rm top} = \tilde{m}$. Fix some $c \in S = \overline{C} \smallsetminus X(d)$ and let $\overline{U}$ be a neighborhood of the cusp, i.e. the spectrum of a discrete valuation ring whose closed point is c. The Néron-Model of the Jacobian J_U of the family of curves $\tilde{f}|_U$ is $J_{\overline{U}} := {\rm Pic}^0_{\mathfrak{X}|_{\overline{U}}/\overline{U}}$. The special fiber J_c of $J_{\overline{U}}$ is an an extension of a semiabelian variety by a finite group, the local component group Φ_{J_c}. By [BLR90] Remark 9.6.12 the cardinality of Φ_{J_c} equals $\tilde{m}$.

We want to compare this to the Néron-Model $E_{\overline{U}}$ of the Jacobian E_U of the family of elliptic curves $h : \mathcal{E} \to \overline{C}$ restricted to U. The component group Φ_{E_c} of the special fiber of $E_{\overline{U}}$ is known to have order d, since we work over $\overline{C} = \overline{X(d)}$.

Recall that A is the fixed part in the family of Jacobians over C. We let $A_{\overline{U}} := A|_U \times_U \overline{U}$. Then the exact sequence of abelian varieties

$$0 \to A|_U \to J_U \to E_U \to 0$$

induces by [BLR90] Proposition 7.5.3 a) an exact sequence

$$0 \to A_{\overline{U}} \to J_{\overline{U}} \to E_{\overline{U}}$$

of Néron models. The right arrow will no longer be surjective.

Since A is constant, we deduce from this sequence an injection $\Phi(\pi_{\mathrm{opt}}) : \Phi_{J_c} \to \Phi_{E_c}$. The cokernel of $\Phi(\pi_{\mathrm{opt}})$ has been determined in [CoSt01] Theorem 6.1: To check their hypothesis, note that π_{opt} is optimal by definition, the degeneration of E_U is toric and Theorem 6.1 in loc. cit. not only works over a p-adic field but over any field complete with respect to a non-archimedian valuation. We introduce some notation to use this result in the form of loc. cit. Corollary 6.6. Let X_{E_c} resp. X_{J_c} be the character group of the toric part of E_c resp. of J_c. There is a natural map $\pi^*_{\mathrm{opt}} X_{\mathcal{E}_c} \to X_{J_c}$ and we let $\mathfrak{L}$ be the saturation of the image. The first quantity we need is $m_{\mathfrak{L}} = [\alpha(X_{J_c}) : \alpha(\mathfrak{L})]$. The homomorphism α is induced by the monodromy pairing. We do not recall details, since in our special situation we need not any property of α other than being a homomorphism. Second, let Θ be the polarization on E_U obtained by pulling back the principal polarization on J_u via $\mathrm{Jac}\,(\pi_{\mathrm{opt}})$. We define $m_{E_U} = \sqrt{\deg(\Theta)}$. Now the formula of [CoSt01] for the size of the cokernel is

$$|\mathrm{coker}(\Phi(\pi_{\mathrm{opt}}))| = m_{E_U}/m_{\mathfrak{L}}$$

In our special case, the intersection graphs $\Gamma_{\mathcal{E}_c}$ and Γ_{J_c} of the components of the singular fibers of both $\mathcal{E}_c$ and J_c are rings by Lemma 4.2. Hence both $H_1(\Gamma_{\mathcal{E}_c}, \mathbb{Z}) \cong X_{\mathcal{E}_c}$ and $H_1(\Gamma_{J_c}, \mathbb{Z}) \cong X_{J_c}$ are cyclic. We conclude that $\mathfrak{L} = X_{J_c}$ and thus $m_{\mathfrak{L}} = 1$.

On the other hand, $m_{E_U} = d_{\mathrm{opt}}$, see e.g. [BiLa03] Lemma 12.3.1. Assembling all the data, we obtain

$$\Delta\chi_{\mathrm{top}} = \tilde{m} = d/|\mathrm{coker}(\Phi(\pi_{\mathrm{opt}}))| = d/d_{\mathrm{opt}},$$

as claimed. □

When we insert the results of the previous lemmas into the Noether formula we obtain:

Corollary 4.15. *For a ST-curve we have*

$$d_{\mathrm{opt}} = \frac{12}{\sum_{i=1}^{s} \frac{k_i^2}{k_i+1} + 16 - 4g}.$$

Hence such curves exist at most for $g = 3$, signature $(1,1,1,1)$ and $d_{\mathrm{opt}} = 2$ – which have been studied in Section 4.2 – or for one of the cases in the following table:

case	*genus*	*signature*	d_{opt}
(1)	*4*	*(1,1,1,1,1,1)*	*4*
(2)	*4*	*(2,2,2)*	*3*
(3)	*5*	*(1,1,1,1,1,1,2)*	*36*
(4)	*5*	*(1,1,1,1,2,2)*	*18*
(5)	*5*	*(1,1,2,2,2)*	*12*
(6)	*5*	*(2,2,2,2)*	*9*
(7)	*5*	*(1,1,1,1,1,3)*	*16*
(8)	*5*	*(1,1,3,3)*	*8*
(9)	*5*	*(1,1,1,1,4)*	*10*

Proof. From Lemma 4.11 and Lemma 4.12 we deduce that on $\widetilde{\mathfrak{X}}_{\mathrm{sec}}$

$$(\omega_{\widetilde{\mathfrak{X}}_{\mathrm{sec}}/\overline{C}_{\mathrm{sec}}})^2 = \left(2(2g-2) - \sum \frac{k_i^2}{k_i+1}\right) \deg \tilde{f}_* \omega_{\widetilde{X_{\mathrm{sec}}}/\overline{C}_{\mathrm{sec}}}.$$

Since $\deg \tilde{f}_* \omega_{\widetilde{X_{\mathrm{sec}}}/\overline{C}_{\mathrm{sec}}} = \deg(\psi) \deg \tilde{f}_* \omega_{\widetilde{X}/\overline{C}} = \deg(\psi) \cdot d\Delta_d$, we have on $\widetilde{\mathfrak{X}}$

$$(\omega_{\widetilde{\mathfrak{X}}/\overline{C}})^2 = \left(2(2g-2) - \sum \frac{k_i^2}{k_i+1}\right) \cdot d\Delta_d.$$

Together with Proposition 4.14 and the Noether formula this yields the claimed expression for d_{opt}.

We now list all the signatures that provide non-negative integer solutions of the equation for d_{opt}, but which are missing in the table: $(4,4)$ and $(1,2,5)$ with $g = 5$ and $(3,7)$, $(1,1,8)$, $(2,8)$, $(1,9)$, $(5,5)$ and (10) with $g = 6$. We argue why they cannot occur for ST-curves.

In the cases with two or less zeros, consider a direction containing a saddle connection joining the two zeros and the stable model of the corresponding degenerate fiber. Since every component of this stable curve is hit by at least one of the sections S_i defined by the zeros and since the one or two zeros in these cases degenerate to the same component by the choice of the direction, the stable curve is irreducible. Lemma 4.2 and the description of the degeneration in Section 2.7 implies that the direction has only one cylinder. This contradicts Lemma 4.5.

In the remaining cases, take a direction containing a saddle connection joining the zero of order 5 (resp. 8) with the zero of order 2 (resp. a zero of order 1). When degenerating in this direction, the remaining zero lies by Lemma 4.4 on the same component as the other two zeros. We now conclude as in the above cases. □

Meanwhile we can significantly strengthen Lemma 4.3. Suppose we factor $\pi = [n] \circ \pi_{\mathrm{opt}}$. We let $E_n = \mathbb{C}/(n\mathbb{Z}[i])$, we denote by $B \subset E_n$ the set of branch points, we let $E_n^* := E_n \setminus B$ and we consider cylinders in E_n^* and in X.

Corollary 4.16. *With the normalization $\pi = [n] \circ \pi_{\mathrm{opt}}$ the width w_θ of each of the cylinders in X is the same in each direction θ.*

In particular, the number of π_{opt}-preimages of a cylinder of E_n^* in the direction θ is independent of the cylinder and independent of θ.

Proof. Suppose that in a fixed direction θ the surface generating the ST-curve has k_θ cylinders C_i of height h_i. Consequently, $d = (\sum_{i=1}^{k_\theta} h_i) \cdot w_\theta$ where w_θ is the width of the cylinder in the direction θ. Let $c = c(\theta)$ be the cusp corresponding to the degeneration in the direction θ. We know by Proposition 4.14 and the normalization of π that the semistable model $\widetilde{\mathfrak{X}}$ of the fiber over c is a ring of $\tilde{m}$ curves, independently of the direction. The stable fiber over c has k_θ components and we have to determine the type of the singularities. The fundamental group locally acts by $\gamma = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$ since we work over the base $\mathbb{H}/\Gamma(d)$. Thus the cylinder C_i is twisted dh_i/w_θ times by γ. This leads to a singularity of type A_{dh_i/w_θ} at the corresponding node of the singular fiber. After resolving by a string of $dh_i/w_\theta - 1$ projective lines the semistable singular fiber is a ring of

$$\tilde{m} = \sum_{i=1}^{k_\theta} \left(h_i \frac{d}{w_\theta} - 1 \right) + k_\theta = \left(\frac{d}{w_\theta} \right)^2$$

curves. Since $\tilde{m}$ is independent of the direction, w_θ is independent, too. □

For each of the cases in Corollary 4.15 we start determinining the number of preimages of cylinders of E_n^* and the location of the branch points B. By the previous corollary, this number is independent of the direction.

Lemma 4.17. *In each of the possible cases of ST-curves listed in Corollary 4.15 one of the following two possibilities holds.*

- i) *The preimage under π_{opt} of each cylinder in E_n^* consists of only one cylinder in X, or*
- ii) *B consists of one element only and each cylinder in E_n^* has k preimages under π_{opt} where $2 \leq k \leq g-1$, or*
- iii) *$g = 5$ and B is the set of 2-torsion points of E_n^*. Moreover, each cylinder in E_n has 2 preimages under π_{opt}.*

Proof. Suppose first that $g = 4$. Suppose there exists some direction θ such that E_n^* decomposes into more than one cylinder in the direction θ. If one cylinder in the direction θ in E_n^* has more than one preimage under π_{opt}, all cylinders in the direction θ have this property by Corollary 4.16. Since the number of horizontal cylinders in X is bounded by $g - 1 = 3$ by Lemma 4.2 and Lemma 4.4, we obtain a contradiction. Corollary 4.16 now implies the same behavior in all the other directions. This gives case i).

The non-existence of a direction θ such that E_n^* decomposes into more than one cylinder in that direction implies that B consists of one element only. The upper bound on k in this case is obtained from the same argument. The lower bound is Lemma 4.5. This concludes case ii).

In the case $g = 5$ we can apply the same argument with one exception: Each cylinder in E_n^* has precisely two preimages in C and each direction of E_n consists of at most two cylinders. The last condition implies that the set of branch points of π_{opt} equals the set of 2-torsion points of E_n. □

We let Ram be the ramification points of π_{opt} and we present the unramified covering

$$\pi_{\text{opt}} : X^* := X \smallsetminus \text{Ram} \to E_n^*$$

by a homomorphism $\varphi_{\text{opt}} : \pi_1(E_n^*, P) \to S_{d_{\text{opt}}}$, where P is some base point. We compose paths from right to left and also compose elements in symmetric groups from right to left.

4.4. **Case d_{opt} is even.** The condition in Lemma 4.17 i) implies that elements of $\pi_1(E^*, P)$ representing core curves of cylinders are mapped to d_{opt}-cycles. The hypothesis d_{opt} is even translates into a d_{opt}-cycle being odd and we thus can argue with parities. We illustrate the basic idea. Suppose that α (resp. β) represents the core curve of a horizontal (resp. vertical) cylinder. Then $\varphi_{\text{opt}}(\alpha)$ and $\varphi_{\text{opt}}(\beta)$ are odd permutations. If none of loops around a branch point was mapped to an odd element in $S_{d_{\text{opt}}}$, then the core curve of a cylinder of slope 1 would have the same parity as $\varphi_{\text{opt}}(\alpha)\varphi_{\text{opt}}(\beta)$, that is even parity. It can thus not correspond to a d_{opt}-cycle and we obtain a contradiction.

We call branch points of π_{opt} whose monodromy is even (resp. odd) in $S_{d_{\text{opt}}}$ *even* (resp. *odd*) *branch points.* They must be in a very special position in order to avoid a contradiction similar to the above one.

Lemma 4.18. *If d_{opt} even and if case i) of Lemma 4.17 holds, then it is possible to normalize E_n by translating the origin such that the set of odd branch points is precisely the set of 2-torsion points of E_n.*

Proof. The set of odd branch points is non-empty by the argument preceding the Lemma and by translation one of them is the origin. By the genus condition of Lemma 4.4 each straight line of rational slope E_n contains an even number of odd branch points. By the list of Corollary 4.15 there are at most 6 odd branch points. From these two statements we deduce that there are precisely 4 odd branch points, given by $P_1 = (0, 0)$, $P_2 = (0, a)$, $P_3 = (a, 0)$ and $P_4 = (a, a)$ for some $a \in \{0, \ldots, n/2\}$.

If $a \neq n/2$, the diagonal of slope -1 contains only one odd branch point, namely P_1. This is a contradiction and completes the claim. □

Corollary 4.19. *There are no ST-curves in the case (1) of Corollary 4.15, i.e. generated by a flat surface in the stratum $\Omega M_4(1, 1, 1, 1, 1, 1)$.*

Proof. For the stratum $\Omega M_4(1, 1, 1, 1, 1, 1)$ the case ii) of Lemma 4.17 cannot appear, since in this case the unique branch point had $12 > d_{\text{opt}} = 4$ preimages (counted with multiplicity).

We now discuss why for this stratum, the case i) of Lemma 4.17 cannot occur. Indeed, since $d_{\text{opt}} = 4$ allows only one ramified point in a fiber over an odd branch point, the only configuration admitted by Lemma 4.18 are 4 odd branch points located at the two-torsion points and another even branch point somewhere else. Using the rotation around P_2 and the reflection along a diagonal we may suppose that P_1 is located in the lower left corner above the diagonal, i.e. in the triangle $P_4P_5P_1$, as indicated in Figure 4. Moreover, if P_1 lies on the boundary of this triangle, we may suppose, applying the shearing by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, if necessary, that P_1 lies on P_2P_4. The presentation of the fundamental group indicated in Figure 4 satisfies the standard relation

$$\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1[\alpha,\beta] = 1.$$

We now list several elements in $\pi_1(E_n^*)$ that are core curves of cylinders in E_n^*.

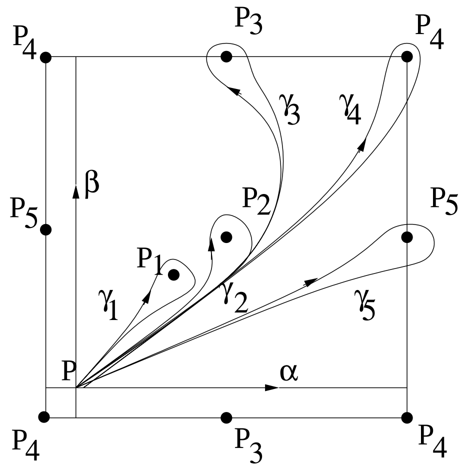

FIGURE 4. Standard presentation of $\pi_1(E_n^*)$ punctured at five points with $P_2, \ldots, P_5$ located at the two-torsion points.

Consequently, all these elements have to map to 4-cycles under $\varphi : \pi_1(E_n^*) \to S_4$. The loops we use are

$$i)\, \alpha, \quad ii)\, \alpha\gamma_1, \quad iii)\, \alpha\gamma_5\gamma_2\gamma_1$$

corresponding to cylinders of slope zero,

$$iv)\, \beta\alpha\gamma_5, \quad v)\, \alpha\beta\gamma_1^{-1}\gamma_2^{-1}\gamma_3^{-1}\gamma_2\gamma_1$$

corresponding to cylinders of slope one,

$$vi)\, \beta, \quad vii)\, \beta\gamma_1^{-1}, \quad vii) \beta\gamma_1^{-1}\gamma_2^{-1}\gamma_3^{-1}$$

corresponding to cylinders of slope ∞ and

$$ix)\, \alpha\gamma_2\gamma_1\beta^{-1}, \quad x)\, \gamma_1^{-1}\alpha\gamma_5\gamma_3\gamma_2\gamma_1\beta^{-1}$$

of slope -1. Relabeling the elements of S_4 we may suppose, without loss of generality, that $\varphi(\alpha) = (1234)$ and that $\varphi(\gamma_5) = (1x)$ for some $x \in \{2,3,4\}$. Cylinder ii) implies that $\gamma_1 = (13)(24)$ and vi) and vii) together imply that $\varphi(\beta) = \varphi(\alpha)^{\pm 1}$. Using iv) only $\varphi(\beta) = \varphi(\alpha)$ is indeed possible and moreover $\varphi(\gamma_5) \in \{(12),(14)\}$. Suppose first that $\varphi(\gamma_5) = (12)$. Then iii) implies $\varphi(\gamma_2) \in \{(12),(23)\}$. We search the two possibilities and all transpositions for $\varphi(\gamma_3)$ for two conditions to hold: The path *vii*) should be mapped to a 4-cycle and $\gamma_4^{-1} = \gamma_3\gamma_2\gamma_1\gamma_5$ has to be mapped to a transposition. The only remaining possibility is $\varphi(\gamma_2) = (14)$, $\varphi(\gamma_3) = (24)$ and $\varphi(\gamma_4) = (23)$. This map φ does not map the cycle x) to a 4-cycle. In the second case $\varphi(\gamma_5) = (14)$ all but the last test are passed only by the possibility $\varphi(\gamma_2) = (34)$, $\varphi(\gamma_3) = (13)$ and $\varphi(\gamma_4) = (12)$. The last test using x) here again fails. □

4.5. **Case** $g = 4$**, signature** $(2,2,2)$**, and** $d_{\mathrm{opt}} = 3$**.** We represent the covering by its monodromy φ_{opt} as in the previous section. $d_{\mathrm{opt}} = 3$ implies together with Lemma 4.17 that there are in fact precisely three branch points. By applying a matrix in $\mathrm{SL}_2(\mathbb{R})$ we may assume that the branch points are located at $P_0 = (0,0)$, $P_1 = (a,0)$ and $P_2 = (0,b)$.

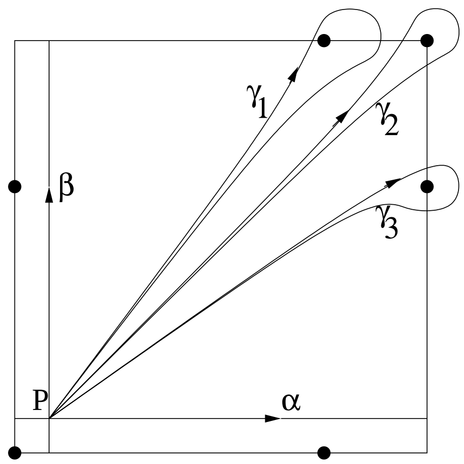

FIGURE 5. Standard presentation of $\pi_1(E_n^*)$ punctured at three points.

Lemma 4.20. *The only ST-curve in the case (2) of Corollary 4.15 is given by* $y^6 = x(x-1)(x-t)$.

Proof: Consider the images of α, β and the γ_i under the monodromy map φ_{opt}. Lemma 4.17 and the hypothesis on the signature imply respectively that they all map to 3-cycles. Since these paths generate $\pi_1(E_n^*, P)$, the image of the map φ_{opt} lies in $\mathbb{Z}/3 = \langle\sigma\rangle \subset S_3$. Since $[\alpha,\beta] = \gamma_1\gamma_2\gamma_3$ we must have $\varphi(\gamma_i) = \sigma$ for $i = 1,2,3$, choosing σ suitably.

By symmetry, we may suppose that $a \geq n/2$ and $b \geq n/2$. Since α and $\alpha\gamma_3$ represent core curves of cylinders, we deduce $\varphi(\alpha) = \sigma$. The same argument applied to β and $\beta\gamma_1^{-1}$ yields $\varphi(\beta) = \sigma^2$. We thus have determined uniquely the branched cover up to the location of a and b. If $a = b = n/2$, e.g. a calculation of the automorphism group shows that the branched cover is given by the curve $y^6 = x(x-1)(x-t)$ (compare [FoMa08]).

If either of a or b is different from $1/2$, we may suppose that $b \neq 1/2$, maybe after flipping along the first diagonal. In this case $\alpha\beta\gamma_1^{-1}\alpha$ also represents a core curve of a cylinder of slope 1/2. Since the φ_{opt}-image of this element is trivial, we obtain the desired contradiction. $\square$

With Corollary 4.19 and Lemma 4.20 we have completed the proof of Theorem 4.1.

In genus 5 the strata where ST-curves can exist, are listed in Corollary 4.15. Moreover, Corollary 4.16 implies strong restrictions to the possible monodromy representations of such a ST-curve in $g = 5$ – if it exists. Whenever d_{opt} is even, i.e. in all strata but the stratum $\Omega M_5(2,2,2,2)$, Lemma 4.18 imposes even further restrictions on the locations of the branch points. We conjecture that there are no ST-curves in $g = 5$.

5. Lyapunov exponents of Teichmüller curves

Fix an $\mathrm{SL}_2(\mathbb{R})$-invariant, ergodic measure μ on ΩM_g^*. The Lyapunov exponents for the Teichmüller geodesic flow on ΩM_g measure the logarithm of the growth rate of the Hodge norm of cohomology classes during parallel transport along the geodesic flow. More precisely, let $\mathbb{V}$ be the restriction of the real Hodge bundle V (i.e. the bundle with fibers $H^1(X,\mathbb{R})$) to the support of μ. Let S_t be the lift of the geodesic flow to V via the Gauss-Manin connection. Then Oseledec's theorem shows the existence of a filtration

$$V = V_{\lambda_1} \supset \cdots V_{\lambda_k} \supset 0$$

by measurable vector subbundles such that, for almost all $m \in M$ and all $v \in V_m \smallsetminus \{0\}$, one has

$$||S_t(v)|| = \exp(\lambda_i t + o(t)),$$

where i is the maximal value such that $v \in (V_i)_m$. The numbers λ_i for $i = 1, \ldots, k \leq \operatorname{rank}(V)$ are called the *Lyapunov exponents of S_t*. Note that these exponents are unchanged if we replace the support of μ by a finite unramified covering with a lift of the flow and the pullback of V. We adopt the convention to repeat the exponents according to the rank of V_i/V_{i+1} such that we will always have $2g$ of them, possibly some of them equal. Since V is symplectic, the spectrum is symmetric, i.e. $\lambda_{g+k} = -\lambda_{g-k+1}$. The reader may consult [Fo06] or [Zo06] for a more detailed introduction to the subject. The guiding questions are existence of zero Lyapunov exponents and simplicity of the set of Lyapunov exponents.

While the Lyapunov spectrum defined is for any $\mathrm{SL}_2(\mathbb{R})$-invariant measure μ, the extremal case that the support of μ is very large, namely a stratum, is best understood. It has been shown by Veech ([Ve86]) that the second Lyapunov exponent is strictly smaller than one. Forni has shown in [Fo02], that $\lambda_g > 0$. Subsequently, Zorich's conjecture, saying that the Lyapunov spectrum is simple, was completed by [AV07].

The analogous statements fail when the support of μ is minimal, i.e. when μ is supported on a closed orbit or equivalently on the natural lift of a Teichmüller curve to ΩM_g. In this case it was discovered by Forni ([Fo06]) that there exists an example with zero Lyapunov exponent and even an example where the second Lyapunov exponent λ_2 is zero, see also below. In this case we say that the Lyapunov spectrum is *totally degenerate*.

We start we a summary of known results on the Lyapunov spectrum for Teichmüller curves.

Theorem 5.1 ([BM05]). *Let $f : X \to C$ be the universal family over a Teichmüller curve with the conventions stated above. Suppose the VHS has a rank two summand $\mathbb{L}$ whose $(1,0)$-part of the Hodge filtration is given by a line bundle $\mathcal{L}$, then the Lyapunov spectrum of the Teichmüller curve is the union of plus and minus*

$$\lambda = \deg \mathcal{L}/(g - 1 + \#\{\overline{C} \setminus C\}/2)$$

and the spectrum of the complement of $\mathbb{L}$.

Sometimes e.g. for cyclic coverings of the projective line, this result allows to calculate all the Lyapunov exponents. See [BM05] Proposition 3.4 for the precise statement. Note that loc. cit. and the above theorem applies only to the summands of the VHS that are non-unitary. For summands of the VHS that are unitary, e.g. those for which the Hodge filtration is trivial, the Lyapunov exponents are zero.

We recall that a family of abelian varieties $g : \mathcal{Y} \to C$ has a fixed part of dimension r, if there is an abelian variety A of dimension r and an injection $A \times C \to \mathcal{Y}$ commuting with the maps of C.

Proposition 5.2. *Suppose that f has a fixed part of dimension r. Then $2r$ of the Lyapunov exponents are zero.*

Proof. Under this assumptions, the variation of Hodge structures splits off a trivial local system $\mathbb{U}$ of rank $2r$. All the Lyapunov exponents corresponding to $\mathbb{U}$ are obviously zero. $\square$

This together with the results of Section 3 imply:

Corollary 5.3 ([Fo06], [FoMa08]). *The Lyapunov spectrum of the family of curves*

$$y^4 = x(x-1)(x-t)$$

resp.

$$y^6 = x(x-1)(x-t)$$

is totally degenerate, that is, it equals

$$\{\pm 1, 0, 0\} \quad \textit{resp.} \quad \{\pm 1, 0, 0, 0\}.$$

Proposition 5.4. *Suppose that the Lyapunov spectrum of a Teichmüller curve f is totally degenerate. Then f is a ST-curve.*

Proof. We write the VHS of f as $\mathbb{V} = \mathbb{L} \oplus \mathbb{M}$, where $\mathbb{L}$ is the maximal Higgs subsystem, i.e. contributes the Lyapunov exponents ± 1 and where $\mathbb{M}$ denotes its orthogonal complement. The hypothesis implies that all the Lyapunov exponents of $\mathbb{M}$ are zero. We claim, that consequently, there is no expansion of the Hodge norm on $\mathbb{M}$ along the geodesic flow. Since the geodesic flow acts ergodically on the unit tangent bundle to C, the claim implies that the local system $\mathbb{M}$ is unitary. Since Teichmüller curves are non-compact, one shows as in [ViZu04] that the splitting of $\mathbb{V}$ is defined over $\mathbb{Q}$, i.e. a splitting of abelian varieties up to isogeny. Since the local system $\mathbb{V}$ carries a $\mathbb{Z}$-structure, $\mathbb{M}$ is local system with transition maps in $\mathrm{Sp}(2g-2, \mathbb{Z}[1/N])$ for some $N \in \mathbb{N}$. The intersection of $\mathrm{Sp}(2g-2, \mathbb{Z}[1/N])$ and the unitary group is obviously finite. Thus, after a finite base change, the family f has a fixed part of dimension $g-1$. Consequently, it is a ST-curve.

To prove the claim, we have to show that the vanishing of the Lyapunov exponents – which a priori implies just subexponential growth – implies that the Hodge norms remain indeed constant. For this purpose we use [Fo02] Corollary 5.3. It states that that

$$\lambda_2 + \cdots \lambda_g = \int_{\mathrm{supp}(\mu)} \Lambda_2(q) + \cdots \Lambda_g(q)\mu(q),$$

where the $\Lambda_i(q) \geq 0$ are the eigenvectors of a hermitian form H_q obtained as a Hodge norm modified using the quadratic differential q (which is $q = \omega^2$ in our case. If $\lambda_i = 0$ for all $i > 1$ then $\Lambda_i = 0$ μ-almost everywhere and for all $i > 1$. By [Fo02] Lemma 2.1, for all cohomology classes c in the orthogonal complement of $\langle \omega, \bar{\omega} \rangle$, the derivative of the Hodge norm is given by

$$\frac{d}{dt}||S_t(c)|| = -2\mathrm{Re}(B_q(S_t(c), S_t(c))),$$

where B_q is a bilinear form with $H_q = B_q^* B_q$. Since H_q vanishes, so does B_q and the derivative. □

We may thus restate the main theorem as follows.

Corollary 5.5. *Up to possible exceptions in M_5 the only Teichmüller curves with totally degenerate Lyapunov spectrum are the ones listed in Corollary 5.3.*

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