An Introduction to Quiver Representations
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1 Algebras, Quivers, Representations, and Categories

1.1 Algebras and Quivers

Question 1.1. Why do we study quivers?

To explain why quivers are a useful object to study from the perspective of algebra, we will need to introduce a new algebraic object called an algebra. Algebras aren't really that new though because they are essentially just vector spaces where you can also multiply the vectors. Throughout the course of these notes, we're going to take $\mathbb{k}$ to be an algebraically closed field. A field $\mathbb{k}$ is **algebraically closed** if when we set any polynomial in one variable whose coefficients are in $\mathbb{k}$ equal to zero, all the solutions live in the field $\mathbb{k}$. We can think of $\mathbb{k}$ as the complex numbers $\mathbb{C}$ if we're familiar with them. If we aren't, we can for the most part think of $\mathbb{R}$ until of course we need the algebraic closure property.

Wait! Why isn't $\mathbb{R}$ algebraically closed?

Definition 1.1.1. A set of objects A with two operations $(+, \cdot)$ is called an (associative) **algebra over** $\mathbb{k}$ with unit 1_A, or an (associative) unital $\mathbb{k}$**-algebra**, if for all $x, y, z \in A$ and $c, d \in \mathbb{k}$, we have the following properties:

1. $x + y \in A$
2. $x + y = y + x$
3. $(x + y) + z = x + (y + z)$
4. There is an element 0 such that $x + 0 = 0 + x = x$
5. For each x there is an element $-x$ such that $x + -x = 0$
6. $cx \in A$
7. $c(x + y) = cx + cy$
8. $(c + d)x = cx + dx$
9. $c(dx) = (cd)x$
10. $1_{\mathbb{k}}x = x$
11. $x \cdot y \in A$
12. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
13. $1_A \cdot x = x \cdot 1_A = x$
14. $x \cdot (y + z) = x \cdot y + x \cdot z$
15. $(x + y) \cdot z = x \cdot z + y \cdot z$
16. $c(x \cdot y) = (cx) \cdot y = x \cdot (cy) = (x \cdot y)c$

Let's unpack this definition a bit. The first 10 axioms should look familiar: they are the axioms of a vector space over the field $\mathbb{k}$. When we are talking about scalar multiplication in the sense of vector spaces, we write the scalar c directly next to the element of the algebra x as in cx. The new things here are axioms 11-16. When we're multiplying two elements of the algebra together, we write $x \cdot y$. This multiplication is *often not the same as the scalar multiplication*! In fact when we talk about quivers, they will be entirely different.

Wait! Is $\mathbb{R}^2$ an algebra? How about $\mathbb{R}^3$?

Exercise 1.1.1. For which values of n is $\mathbb{R}^n$ an algebra? For these value(s), find a field k and show that $\mathbb{R}^n$ satisfies the axioms of an associative k-algebra with unit.

We know from linear algebra that a **basis of a vector space** V is a linearly independent set of vectors that span V and that the dimension of V is the number of vectors in a basis. If we're rusty let's dust off the old linear algebra book and refresh! Remember that algebras are also vector spaces, so we can define the **dimension of the algebra** A as its dimension as a vector space.

Example 1.1.1. Consider the set of $n \times n$ matrices such that all the entries below the diagonal are 0. We call these matrices **upper triangular**. In the 3×3 case they look like this:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}.$$

We can add these together and get another upper triangular matrix. We can multiply them and get an upper triangular matrix. In fact, the collection of these matrices forms a $\Bbbk$-algebra denoted by $U_n(\Bbbk)$.

Exercise 1.1.2. Show that $U_n(\Bbbk)$ is an associative $\Bbbk$-algebra with unit.

We will denote by $M_{ij} \in U_n(\Bbbk)$, where $i \leq j$, the matrix that has 1 in the ijth spot and zeros everywhere else. So in the 3×3 case, M_{12} is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

One can (and should) show that the collection of these M_{ij} form a basis for $U_n(\Bbbk)$ and in general, there are $\frac{n(n+1)}{2}$ of these basis elements so the algebra is finite dimensional. Now let's make a **multiplication table** for the algebra $U_n(\Bbbk)$. We start with one column and one row where each row and column corresponds to a basis element for the algebra.

	M_{11}	M_{22}	M_{33}	M_{12}	M_{23}	M_{13}
M_{11}						
M_{22}						
M_{33}						
M_{12}						
M_{23}						
M_{13}						

To find the matrix that needs to fill the first empty space, we multiply the matrix corresponding to the row on the left by the matrix corresponding to the column on the right. Notice that the order in which we multiply *is very important* because in general, multiplication in an algebra *is not commutative*. So the top left entry is $M_{11} \cdot M_{11} = M_{11}$. The entry immediately to its right is $M_{11} \cdot M_{22} = 0$ where 0 is the zero matrix. Continuing in this way, we get the full multiplication table (check it!):

	M_{11}	M_{22}	M_{33}	M_{12}	M_{23}	M_{13}
M_{11}	M_{11}	0	0	M_{12}	0	M_{13}
M_{22}	0	M_{22}	0	0	M_{23}	0
M_{33}	0	0	M_{33}	0	0	0
M_{12}	0	M_{12}	0	0	M_{13}	0
M_{23}	0	0	M_{23}	0	0	0
M_{13}	0	0	M_{13}	0	0	0

Wait! Why is $\{M_{11}, M_{22}, M_{33}, M_{12}, M_{23}, M_{13}\}$ a basis if $M_{12} \cdot M_{23} = M_{13}$?

Example 1.1.2. In this example, let's consider the collection of all polynomials of any degree with coefficients in $\Bbbk$, denoted by $\Bbbk[x]$. Again, we can add these together and get another polynomial with coefficients in $\Bbbk$. Given two polynomials in $\Bbbk[x]$, for example $ax + b$ and $cx + d$, we can multiply them to get $acx^2 + (bc + ad)x + bd$, which is another polynomial with coefficients in $\Bbbk[x]$.

Exercise 1.1.3. Show that $\Bbbk[x]$ is an associative unital $\Bbbk$-algebra.

Now let's think of a basis for the vector space. An arbitrary (random) polynomial in $\Bbbk[x]$ is of the form $ax^n + bx^{n-1} + \cdots + yx + z$. To build this out of linear combinations of small pieces, we need 1 to get the constant term z. We also need x to get the term yx. In fact, we need x^m for all m to build an arbitrary polynomial in $\Bbbk[x]$. Therefore a basis would be $\{1, x, x^2, \ldots\}$. This is an example of an infinite dimensional algebra.

Okay, enough with the abstract nonsense. Let's see how we can relate this to quivers.

Definition 1.1.2. A **quiver** is a quadruple (Q_0, Q_1, s, t) where Q_0 is a set of vertices, Q_1 is a set of arrows, $s : Q_1 \to Q_0$ is a function that gives the starting point of each arrow, and $t : Q_1 \to Q_0$ is a function that gives the end point of each arrow.

That definition kind of stinks because it is really notation heavy. A quiver is nothing more than a directed graph:

Example 1.1.3. Let Q be the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. Then the set of vertices is $Q_0 = \{1, 2, 3\}$, the set of arrow is $Q_1 = \{\alpha, \beta\}$, the starting points are $s(\alpha) = 1$, $s(\beta) = 2$, and the terminal points are $t(\alpha) = 2$, $t(\beta) = 3$.

Example 1.1.4. Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[loop right, "\alpha"] \end{tikzcd}$$

Then $Q_0 = \{1\}, Q_1 = \{\alpha\}$, and $s(\alpha) = t(\alpha) = 1$.

Definition 1.1.3. A **path** in the quiver Q is a sequence of arrows $\alpha_1 \alpha_2 \ldots \alpha_n$ such that for each i, $s(\alpha_{i+1}) = t(\alpha_i)$.

Let's look back at the quiver in Example 1.1.3. Pretend like we are standing at vertex 1 and we need to walk to vertex 3. Then the path we need to take is first we walk along α to get to vertex 2. Then since β starts where α ends, we can walk along β to get to vertex 3. The corresponding path would be $\alpha\beta$. A word of caution: since at this point the reader is *so excited* about quivers that they will likely go read other sources about them, we should mention that some people would denote this path as $\beta\alpha$. If we think of paths like composing functions then this convention makes sense, but it is *not* the convention that we will use. Now what if we start at vertex i and we don't move? Ah yes, now we're doing philosophy and not math! Well, in this case we have followed the **lazy path** or the **trivial path**, denoted by e_i. Surprisingly, these paths are quite important as we will soon see!

We said in the beginning that in order to motivate *why* quivers, we needed to introduce some abstract complicated notion of associative unital $\Bbbk$-algebras whose definition relies on 16 different axioms. How on earth could this be connected to quivers whose definition is three words: 'a directed graph'? Well it turns out that to each quiver, we can associate an associative unital algebra!

Definition 1.1.4. To each quiver $Q = (Q_0, Q_1, s, t)$, we define the **path algebra** $\Bbbk Q$ as the algebra whose $\Bbbk$-vector space basis is the set of all paths in the quiver. We define multiplication of two paths p and p' by

$$p \cdot p' = \begin{cases} pp' & t(p) = s(p') \\ 0 & \text{otherwise} \end{cases}$$

In the above definition, when we write the juxtaposition pp', we mean the concatenation of paths p and p'; that is, do path p first, then do path p' immediately afterward. Also, hidden in the notation here is the fact that we haven't defined $s(p)$. A path $p = \alpha_1\alpha_2 \ldots \alpha_n$, so by $s(p)$ we mean $s(\alpha_1)$ and $t(p) = t(\alpha_n)$; that is, where the path starts and where the path ends respectively. Let's see some examples.

Example 1.1.5. Let Q be the quiver from Example 1.1.3. Then the basis for $\Bbbk Q$ is the set of all paths $\{e_1, e_2, e_3, \alpha, \beta, \alpha\beta\}$ (don't forget the lazy paths!). Since we have a basis, an arbitrary element of $\Bbbk Q$ will be of the form $a_1e_1 + a_2e_2 + a_3e_3 + a_4\alpha + a_5\beta + a_6\alpha\beta$ where the $a_i \in \Bbbk$ are scalars. These linear combinations don't have a physical interpretation, they are just formal objects that frankly, we don't use very often unless we're proving things. On the other hand, multiplication has a nice physical interpretation. If we start at vertex 1, we can stand there and do nothing, then follow α to vertex 2 and the resulting path is just α. This corresponds to the product $e_1 \cdot \alpha = \alpha$. But if we replaced e_1 with e_2, then we can't stand at vertex 2 and do nothing, then follow α because α starts at vertex 1. This means $e_2 \cdot \alpha = 0$. On the other hand, we can follow α then stand at vertex 2 and do nothing, so $\alpha \cdot e_2 = \alpha$. Notice that $e_2 \cdot \alpha \neq \alpha \cdot e_2$, so multiplication in these path algebras is rarely commutative. Okay, let's make the multiplication table the same way we did in Example 1.1.1. Looks familiar doesn't it?

	e_1	e_2	e_3	α	β	$\alpha\beta$
e_1	e_1	0	0	α	0	$\alpha\beta$
e_2	0	e_2	0	0	β	0
e_3	0	0	e_3	0	0	0
α	0	α	0	0	$\alpha\beta$	0
β	0	0	β	0	0	0
$\alpha\beta$	0	0	$\alpha\beta$	0	0	0

It turns out that the path algebra in Example 1.1.3 is 'the same' as the one in Example 1.1.1 as the reader may have already guessed since they have the exact same multiplication table up to 'renaming'. To make precise what we mean by 'the same', we need to introduce another definition.

Definition 1.1.5.

- A **homomorphism** between two $\Bbbk$-algebras A_1 and A_2 is a function $f : A_1 \to A_2$ such that for any $\alpha, \beta \in A_1$ and $c \in \Bbbk$ we have the following:

    1. $f(\alpha + \beta) = f(\alpha) + f(\beta)$
    2. $f(\alpha \cdot \beta) = f(\alpha) \cdot f(\beta)$
    3. $f(c\alpha) = cf(\alpha)$

- A homomorphism is called an **isomorphism** if it is bijective. Two algebras are **isomorphic** if there exists an isomorphism between them. We denote this by $A_1 \cong A_2$.

Wait! Is the plus sign on the left hand side of equality 1 the same as the one on the right? How about the $\cdot$ in equality 2?

Notice that a function between two vector spaces that satisfies properties 1 and 3 in the definition of an algebra homomorphism is called a linear transformation! It turns out we actually think of these things as morphisms in the 'category of vector spaces' which we will define later. So really the only new thing here is the linearity of f with respect to the algebra multiplication (property 2). The notion of two $\Bbbk$-algebras being 'the same' is captured by the notion of isomorphism and the act of 'renaming' is formalized by homomorphisms.

Proposition 1.1.1. *The algebra $U_3(\Bbbk)$ is isomorphic to $\Bbbk Q$, the path algebra of the quiver in Example 1.1.3.*

Proof. We need to construct an isomorphism between the two algebras. Actually, since an arbitrary element of $\Bbbk Q$ looks like $a_1e_1 + a_2e_2 + a_3e_3 + a_4\alpha + a_5\beta + a_6\alpha\beta$, it suffices to decide where the basis elements go then extend f to arbitrary elements of $\Bbbk Q$ linearly (ie force f to satisfy properties 1-3 in Definition 1.1.5). We can do this by looking at the multiplication table. Define $f : \Bbbk Q \to U_3(\Bbbk)$ by

$$\begin{array}{ll} e_1 \mapsto M_{11} & \alpha \mapsto M_{12} \\ e_2 \mapsto M_{22} & \beta \mapsto M_{23} \\ e_3 \mapsto M_{33} & \alpha\beta \mapsto M_{13} \end{array}$$

Check that this assignment makes sense! Since $\alpha \cdot \beta = \alpha\beta$, we need $M_{12} \cdot M_{23} = M_{13}$ and so on. After extending to the entire algebra linearly, we get a bijective algebra homomorphism, and therefore an isomorphism. So the two algebras are actually the same after all! □

Exercise 1.1.4. Show that $U_n(\Bbbk) \cong \Bbbk Q$ for $Q = 1 \to 2 \to \cdots \to n-1 \to n$. (Hint: look back at the proof for the case $n = 3$. Which matrix did the path from 1 to 2 go? How about the path from 2 to 3 or 1 to 3?)

Proposition 1.1.2. *Let Q be the one loop quiver from Example 1.1.4. Then as algebras, $\Bbbk Q \cong \Bbbk[x]$ where $\Bbbk[x]$ is the polynomial algebra from Example 1.1.2.*

Exercise 1.1.5. Prove the previous proposition by providing an explicit isomorphism.

Okay, so we said that to each quiver, we can associate a unital algebra called the path algebra, but we haven't discussed what the unit is! Let's look back at $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. Recall that $\Bbbk Q \overset{f}{\cong} U_3(\Bbbk)$. We know the unit in $U_3(\Bbbk)$ is the 3×3 identity matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To figure out what the unit in $\Bbbk Q$ is, we just need to find the element in $\Bbbk Q$ that gets send to I_3 by f, or equivalently, we need to find $f^{-1}(I_3)$.

Wait! Why is it true that an isomorphism of algebras send the unit to the unit?

To figure this out, we know that $I_3 = M_{11} + M_{22} + M_{33}$. Thus

$$\begin{aligned} f^{-1}(I_3) &= f^{-1}(M_{11} + M_{22} + M_{33}) \\ &= f^{-1}(M_{11}) + f^{-1}(M_{22}) + f^{-1}(M_{33}) (\text{Why?}) \\ &= e_1 + e_2 + e_3. \end{aligned}$$

Therefore, the unit in $\Bbbk Q$ is the sum of the lazy paths! See, we said they would be important.

Proposition 1.1.3. *Let Q be a quiver and $\Bbbk Q$ its path algebra. Then $1 \in \Bbbk Q$ is given by the sum of the lazy paths $\displaystyle\sum_{i \in Q_0} e_i$.*

Proof. Suppose that $|Q_0| = n$, so Q has n vertices and let $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a basis for $\Bbbk Q$. Then an arbitrary element of $a \in \Bbbk Q$ is of the form $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m$ for scalars $c_i \in \Bbbk$. Notice that any path starts at one and only one vertex, so we can split this sum up into the paths that start at vertex 1, those that start at vertex 2, up to those that start at n. To do this, we write $a = (c_1\alpha_1 + c_2\alpha_2 + \cdots + c_{i_1}\alpha_{i_1}) + \cdots + (c_{i_{n-1}+1}\alpha_{i_{n-1}+1} + \cdots + c_m\alpha_m)$.

We need to first show that $(e_1 + e_2 + \cdots + e_n) \cdot a = a$. We have

$$\begin{aligned}
\sum_{i \in Q_0} e_i \cdot a &= \sum_{i \in Q_0} e_i \cdot (c_1\alpha_1 + c_2\alpha_2 + \cdots + c_{i_1}\alpha_{i_1}) + \cdots + (c_{i_{n-1}+1}\alpha_{i_{n-1}+1} + \cdots + c_m\alpha_m) \\
&= e_1 \cdot (c_1\alpha_1 + c_2\alpha_2 + \cdots + c_{i_1}\alpha_{i_1}) + \cdots + e_n \cdot (c_{i_{n-1}+1}\alpha_{i_{n-1}+1} + \cdots + c_m\alpha_m) \\
&= (c_1\alpha_1 + c_2\alpha_2 + \cdots + c_{i_1}\alpha_{i_1}) + \cdots + (c_{i_{n-1}+1}\alpha_{i_{n-1}+1} + \cdots + c_m\alpha_m) \\
&= a
\end{aligned}$$

The second equality above holds because $e_i \cdot p = 0$ for any path p that does not start at i and the third equality holds because $e_i \cdot p = p$ for any path p that *does* start at i. A similar argument (write it down!) shows that $a \cdot \sum_{i \in |Q_0|} e_i = a$, which proves the statement. □

Okay great, we have shown that to each quiver, we get a unital associative algebra called the path algebra. So quivers tell us about certain algebras, but do they tell us about all algebras? Well, unfortunately the answer to this question is no, but it is almost yes. To each 'basic' algebra, we can associate a quiver and each path algebra of a quiver is 'basic'. So essentially, studying quivers is the same thing as studying 'basic' associative unital $\Bbbk$-algebras... quite the mouthful. Even though not every algebra is 'basic', we can associate a 'basic' algebra to every algebra, and this associated 'basic' algebra retains many of the qualities of the algebra that people care about (the structure of the module category). So in short, studying quivers and their 'representations' is the same thing as studying 'modules over basic associative unital algebras'. If none of that is meaningful to the reader, we can forget about it as it won't be important moving forward. It is just a way to show that quivers are useful in the general study of representation theory of algebras.

1.2 Representations of Quivers and Morphisms Between Them

Question 1.2. What is the main object of study in this class?

Since this class is called quiver representations, we should probably define a quiver representation.

Definition 1.2.1.

- Let $Q = (Q_0, Q_1, s, t)$ be a quiver with n vertices. A **$\Bbbk$-representation** $V = (V_i, \phi_\alpha)$ of Q is an assignment of a $\Bbbk$-vector space V_i for each $i \in Q_0$ and a linear transformation $\phi_\alpha : V_i \to V_j$ for each arrow $\alpha \in Q_1$ that starts at i and ends at j.

- A representation is called **finite dimensional** if and only if all the vector spaces are finite dimensional.

- The **dimension vector** of a finite dimensional representation V is the vector $\mathbf{dim}(V) = (\dim(V_1), \dim(V_2), \ldots, \dim(V_n))$ that contains the dimension of the ith vector space in the ith position.

- An **element** v of the representation V, is an n-tuple $v = (v_1, v_2, \ldots, v_n)$ where $v_i \in V_i$ for all i.

Another notation heavy definition, so let's see some examples.

Example 1.2.1. Let $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. Then to make a representation, let's pick some finite dimensional vector spaces for vertices 1,2, and 3, say $\Bbbk, \Bbbk^3, \Bbbk^2$ respectively. Think of $\mathbb{R}, \mathbb{R}^3$, and $\mathbb{R}^2$ for now if that helps. Now we need to pick some linear transformations (only two since there are only two arrows in Q). Define $\phi_\alpha : \Bbbk \to \Bbbk^3$ to be the linear transformation whose standard matrix is given by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. We can similarly pick $\phi_\beta : \Bbbk^3 \to \Bbbk^2$ to have the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then this gives the following representation whose dimension vector is $(1, 3, 2)$:

$$V : \Bbbk \xrightarrow{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \Bbbk^3 \xrightarrow{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \Bbbk^2$$

Another representation, whose dimension vector is $(1, 1, 0)$, is $W : \Bbbk \xrightarrow{1} \Bbbk \xrightarrow{0} 0$. The reader should probably draw some more on their own now.

Example 1.2.2. Let's consider a different quiver now. Let Q be the quiver

$$\begin{tikzcd} & & 3 \arrow[dl, "\beta"'] \\ 1 \arrow[r, "\alpha"] & 2 \arrow[dr, "\gamma"] & \\ & & 4 \end{tikzcd}$$

Then the following are examples of representations:

$$\begin{tikzcd} & & 0 \arrow[dl, "0"'] \\ V : \Bbbk \arrow[r, "1"] & \Bbbk \arrow[dr, "1"] & \\ & & \Bbbk \end{tikzcd}$$

$$\begin{tikzcd} & & \Bbbk \arrow[dl, "1"'] \\ W : 0 \arrow[r, "0"] & \Bbbk \arrow[dr, "1"] & \\ & & \Bbbk \end{tikzcd}$$

Zooming Out 1. *Recall from the previous section, we defined algebras, then defined the homomorphisms between them. We've defined representations of quivers, and now it's time to define morphisms between them. In general, once we've defined a class of objects, like vector spaces, algebras, or representations, the next step is to define the morphisms between them. Loosely speaking, a class of objects along with a class of morphisms is a* ***category****.*

Definition 1.2.2. Let $V = (V_i, \phi_\alpha)$ and $W = (W_j, \psi_\alpha)$ be two representations of some quiver Q with n vertices.

- A **morphism** from V to W, denoted by $f = (f_1, f_2, \dots f_n)$, is an n-tuple of linear transformations $f_i : V_i \to W_i$ such that $\psi_\beta \circ f_i = f_{i+1} \circ \phi_\beta$ for each arrow β from i to j in Q.
- A morphism is called **injective** (**surjective**) if f_i is injective (surjective) for each i.
- A morphism is called an **isomorphism** if f_i is a bijection for each i.
- Two representations V and W are called **isomorphic**, denoted $V \cong W$, if there is an isomorphism between them.
- The class of all representations isomorphic to V is called the **isoclass** of V.

The fact that $\psi_\beta \circ f_i = \phi_\beta \circ f_{i+1}$ is often called a *commutivity condition*. To see why, it will probably help to draw some pictures.

Example 1.2.3. Let Q be the quiver $1 \to 2 \to 3$. Then we have two representations given by V and W in Example 1.2.1. A morphism is given by the following:

$$\begin{tikzcd}
V : \Bbbk \arrow[r, "\begin{bmatrix}1\\0\\0\end{bmatrix}"] \arrow[d, "1"'] & \Bbbk^3 \arrow[r, "\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}"] \arrow[d, "\begin{bmatrix}1&0&0\end{bmatrix}"] & \Bbbk^2 \arrow[d, "0"] \\
W : \Bbbk \arrow[r, "1"] & \Bbbk \arrow[r, "0"] & 0
\end{tikzcd}$$

In this example, the morphism is given in blue by $f = (1, \begin{bmatrix}1 & 0 & 0\end{bmatrix}, 0)$. The commutivity relations in Definition 1.2.2 can be seen by noticing that 'all the boxes commute'. If we start at the top left vertex above, go down then right; that must be the same thing as starting at the top left vertex, going right then going down. Mathematically speaking, suppose we have an element $x \in \Bbbk$ in the top left vertex. Then $\psi_\alpha(f_1(x)) = \psi_\alpha(x) = x$. Similarly, $f_2(\phi_\alpha(x)) = f_2\left(\begin{bmatrix}x\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0\end{bmatrix} \cdot \begin{bmatrix}x\\0\\0\end{bmatrix} = x$. Therefore $\psi_\alpha \circ f_1 = f_2 \circ \phi_\alpha$, and the first box commutes. One can (and should) show that the second box also commutes.

Wait! If we picked $f_2 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}$, would this be a morphism?

Example 1.2.4. Now let Q be the quiver from Example 1.2.2. Then $f = (1, 1, 0, 0)$ is a morphism from V to W. It is is shown in blue and the reader should verify that the *three* boxes do indeed commute.

$$
\begin{tikzcd}
 & & & 0 \arrow[dl, "0"'] \arrow[ddd, bend left, "0"] \\
V: & \Bbbk \arrow[r, "1"] \arrow[ddd, "1"] & \Bbbk \arrow[dr, "1"] \arrow[ddd, "1"] & \\
 & & & \Bbbk \arrow[ddd, bend right, "0"'] \\
 & & & \Bbbk \arrow[dl, "1"'] \\
W: & \Bbbk \arrow[r, "1"] & \Bbbk \arrow[dr, "0"] & \\
 & & & 0
\end{tikzcd}
$$

Now that we have the notion of morphisms, we should study them more deeply. One can ask questions like: How many are there? The answer to this is there can be infinitely many! A follow up question may be: In the case when there are infinitely many, can we describe all the morphisms using only a finite collection of them? This idea of building things out of smaller pieces should be familiar: it is precisely the idea behind the basis of a vector space!

Proposition 1.2.1. *Let $V = (V_i, \phi_\alpha)$ and $W = (W_i, \psi_\alpha)$ be two $\Bbbk$-representations of a quiver Q. Then the collection of all morphisms from V to W, denoted $\mathrm{Hom}(V, W)$, is a $\Bbbk$-vector space.*

Proof. In order to show that $\mathrm{Hom}(V, W)$ is a $\Bbbk$ vector space, we need to establish addition of two morphisms and multiplication of a morphism by a scalar in $\Bbbk$. Let $f, g \in \mathrm{Hom}(V, W)$, so $f, g : V \to W$, and $v \in V$ be an element of the representation. We need to define the sum of these two morphisms, and show that it is indeed a morphism from V to W. Define $f + g : V \to W$ by $f + g(v) = f(v) + g(v)$. At vertex i, we have $(f + g)_i(v_i) = f_i(v_i) + g_i(v_i)$. Since both $f_i(v_i)$ and $g_i(v_i)$ are elements of the vector space W_i, so is their sum since vector spaces are closed under addition. Since i was arbitrary, this holds for all vertices i and hence $f(v)+g(v)$ is an element of W. Now we also need to show that $f+g$ satisfies the commutivity relation. Let $\alpha \in Q_1$ be an arrow from vertex i to vertex j. Then

$$
\begin{aligned}
(f+g)_{i+1}(\phi_\alpha(v_i)) &= f_{i+1}(\phi_\alpha(v_i)) + g_{i+1}(\phi_\alpha(v_i)) && \text{(by definition)} \\
&= \psi_\alpha(f_i(v_i)) + \psi_\alpha(g_i(v_i)) && \text{(f and g satisfy commutivity relation)} \\
&= \psi_\alpha((f+g)_i(v_i))
\end{aligned}
$$

Again, since i was arbitrary, we have shown that $f + g \circ \phi_\alpha = \psi_\alpha \circ f + g$. Therefore $f + g$ is a well defined morphism from V to W.

Now let $c \in \Bbbk$ be a scalar. Then define $cf : V \to W$ by $v \mapsto c(f(v))$. Then one can (and should) show that this is also a well-defined morphism from V to W.

Exercise 1.2.1. Finish the proof; that is, show that $\mathrm{Hom}(V, W)$ satisfies the remaining vector space axioms.

$\square$

Okay, let's compute some Hom spaces.

Example 1.2.5. Let Q, V, and W be as in Example 1.2.4. We will compute $\mathrm{Hom}(V, W)$. Any $f \in \mathrm{Hom}(V, W)$ will be of the form $f = ([a], [b], [c], [d])$ where the matrix with one element $[a]$ corresponds to multiplication by a:

$$\begin{tikzcd}
& & & 0 \arrow[dl, "0"'] \arrow[ddd, bend left, "{[d]}"] \\
V: & \Bbbk \arrow[r, "1"] \arrow[ddd, "{[a]}"'] & \Bbbk \arrow[dr, "1"] \arrow[ddd, "{[b]}"'] & \\
& & & \Bbbk \arrow[ddd, bend right, "{[c]}"'] \\
& & & \Bbbk \arrow[dl, "1"'] \\
W: & \Bbbk \arrow[r, "1"] & \Bbbk \arrow[dr, "0"] & \\
& & & 0
\end{tikzcd}$$

The morphism f must satisfy the commutivity relations of each box. Since $[c]$ is a map from $\Bbbk$ to the zero vector space, $[c]$ must be the zero map. Similarly, $[d]$ must also be the zero map. Let's think about why this is. Consider the vector space V and a linear transformation $f : V \to 0$. Then in order for f to be a well-defined linear transformation, we need that $f(x) \in 0$ for each $x \in V$; that is, we require that $f(x) = 0$ for all $x \in V$. This forces f to be the zero map.

Conversely, recall that two functions f and g are equal if they have the same domain and codomain, and they agree on every element of the domain, that is $f, g : X \to Y$ and $f(x) = g(x)$ for all $x \in X$. Okay great! Now let $f : 0 \to V$ be any linear transformation. Then we have that $f(0) = 0$ (why?). We also have the zero map $0 : 0 \to V$ that sends $0 \mapsto 0(0) = 0$. But this means that the linear transformation f has the same domain and codomain as the zero map and $f(x) = 0(x)$ for each $x \in 0$ (since there is only one, namely $x = 0$). What we just showed is that all maps out of the zero space are equal to the zero map. This makes the zero space both an initial and terminal object (and hence the zero object) in the category of vector spaces... whatever that means.

Anyway, this means that $f = ([a], [b], 0, 0)$. Let's look now at the front left box. Let $x \in \Bbbk$ and starting at the top left corner and moving right, we have $x \mapsto 1(x) \mapsto bx$. Now going down first, we have $x \mapsto ax \mapsto ax$. In order for this box to commute, we must have $ax = bx$, which occurs if and only if $a = b$. Therefore, we conclude that $f = ([a], [a], 0, 0)$. (The reader should check that the two boxes on the right do not tell us anything about $[b]$). Since any f is of this form, we conclude that a basis for $\mathrm{Hom}(V, W) = \{(1, 1, 0, 0)\}$, since multiplying this vector by the scalar a will give us the general form for any morphism from V to W. Therefore $\mathrm{Hom}(V, W) \cong \Bbbk$ is one dimensional.

Example 1.2.6. Let's now take Q to be the Kronecker quiver $1 \rightrightarrows 2$ and consider two representations V and W shown below. We will compute $\mathrm{Hom}(V, W)$.

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
V: \Bbbk \arrow[r, shift left, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] \arrow[r, shift right, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"'] \arrow[d, "\begin{bmatrix} a \\ b \end{bmatrix}"'] & \Bbbk^2 \arrow[d, "\begin{bmatrix} c & d \\ e & f \\ g & h \end{bmatrix}"] \\
W: \Bbbk^2 \arrow[r, shift left, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}"] \arrow[r, shift right, "\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}"'] & \Bbbk^3
\end{tikzcd}
$$

In this figure, there are two commuting squares. We either follow the top arrow in both representations, or we follow the bottom arrow in both.

Wait! Why aren't there four squares?

Let $x \in \Bbbk$. Starting in the top left and following the top arrows, we have that

$$x \mapsto \begin{bmatrix} x \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} cx \\ ex \\ gx \end{bmatrix}.$$

Following the other side of the same square,

$$x \mapsto \begin{bmatrix} ax \\ bx \end{bmatrix} \mapsto \begin{bmatrix} ax \\ bx \\ 0 \end{bmatrix}.$$

Commutivity of the square forces $a = c$, $b = e$, and $g = 0$. Following the other square we get

$$x \mapsto \begin{bmatrix} 0 \\ x \end{bmatrix} \mapsto \begin{bmatrix} dx \\ fx \\ hx \end{bmatrix}$$

and

$$x \mapsto \begin{bmatrix} ax \\ bx \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ ax \\ bx \end{bmatrix}.$$

By commutivity, we need $d = 0$, $a = f$, and $b = h$. Therefore, an arbitrary $f \in \mathrm{Hom}(V, W)$ will be of the form

$$f = \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a & 0 \\ b & a \\ 0 & b \end{bmatrix} \right).$$

A basis for this Hom space would be

$$\left\{ \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right), \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right\}$$

since a linear combination of a times the first plus b times the second will give us an arbitrary morphism in the Hom space. Therefore $\mathrm{Hom}(V, W) \cong \Bbbk^2$ is two dimensional.

Now that we're starting to get a handle on morphisms, let's finish this section by formally defining a category.

Zooming Out 2. *Let $\mathscr{C}$ be a class of **objects** denoted by $Ob(\mathscr{C})$, and a class of **morphisms** between the objects denoted by $Hom_{\mathscr{C}}$, such that each morphism $f \in Hom_{\mathscr{C}}$ has a unique source X and target Y. (Think of objects as vertices and morphisms as arrows between them)! In this case, f is **a morphism from** X **to** Y, written $f : X \to Y$. The class of all morphisms from X to Y is denoted by $Hom_{\mathscr{C}}(X, Y)$, and may not be a vector space like we've seen so far.*

*Now $\mathscr{C}$ is called a **category** if we can compose morphisms 'nicely'. More formally, by 'nicely', we mean that for every three objects $X, Y, Z \in Ob(\mathscr{C})$, there is a binary operation $Hom_{\mathscr{C}}(X, Y) \times Hom_{\mathscr{C}}(Y, Z) \to Hom_{\mathscr{C}}(X, Z)$ called composition, sending $(f, g) \mapsto f \circ g$ that satisfies the following.*

1. *If $f : W \to X, g : X \to Y$, and $h : Y \to Z$ are morphisms, then the composition is **associative**:*
$$h \circ (g \circ f) = (h \circ g) \circ f.$$

2. *For every object X there exists a morphism $1_X \in Hom_{\mathscr{C}}(X, X)$ called **the identity morphism on** X such that for every $f : X \to Y$ and $g : Z \to X$, we have*
$$f \circ 1_X = f \text{ and } 1_X \circ g = g.$$

Example 1.2.7.

1. The category of finite-dimensional $\Bbbk$-vector spaces and linear transformations between them.

2. The category of finite-dimensional representations of a quiver Q, denoted by rep$Q$, with morphisms of representations.

3. A quiver! The vertices are the objects and the arrows are the morphisms (what is the identity morphism on each vertex?).

Exercise 1.2.2. Prove that the categories in Example 1.2.7 are indeed categories.

1.3 Direct sums, Indecomposables, Subrepresentations, and Kernels

Question 1.3. How can we use what we know about representations to build new ones?

Using what we know to build new mathematical objects is a common idea in mathematics. In particular, just like with vector spaces, if we have two representations V and W of a quiver Q, we can take their direct sum to get a new, in some sense bigger, representation. In the world of finite-dimensional vector spaces, this is the same thing as taking the product (up to isomorphism).

Definition 1.3.1. Let $V = (V_i, \phi_\alpha)$ and $W = (W_i, \psi_\alpha)$ be two representations of Q. We define the representation known as the **direct sum** of V and W as

$$V \oplus W = \left(V_i \oplus W_i, \begin{bmatrix} \phi_\alpha & 0 \\ 0 & \psi_\alpha \end{bmatrix} \right).$$

The notation here for the morphisms may be a little strange; it is written in block notation! Recall that we can think of the direct sum of vector spaces $V_i \oplus W_i$ in two ways. One way, likely the more familiar way, is as an external direct sum: $V_i \oplus W_i := \{(v_i, w_i) : v_i \in V_i \text{ and } w_i \in W_i\}$. Another way to think about the direct sum is as an internal direct sum. We think of V_i and W_i as two subspaces of $V_i \oplus W_i$ whose intersection is only the zero vector and whose external direct sum is all of $V_i \oplus W_i$. In this case, each element of $V_i \oplus W_i$ can be uniquely expressed as a sum of two vectors, one from V_i and one from W_i. In light of this information, the internal direct sum is $V_i \oplus W_i := \{v_i + w_i : v_i \in V_i \text{ and } w_i \in W_i\}$. It is not an abuse of notation that we use the same symbol $V_i \oplus W_i$ for both the external and internal sums!

Exercise 1.3.1. Given two finite dimensional $\Bbbk$-vector spaces, show that the internal direct sum is isomorphic to the external direct sum.

As a result of Exercise 1.3.1, we can think of elements of the representation $V \oplus W$ as either ordered pairs (v, w) or sums $v + w$ where v is an element of the representation V and w is an element of the representation W. Since $V \oplus W$ is in fact a representation, we can take its direct sum with the representation X. The resulting representation is $(V \oplus W) \oplus X = V \oplus W \oplus X$. In fact, for any m objects in rep$Q$, say $V_1, V_2, \ldots, V_m$, we can define their direct sum recursively as

$$V_1 \oplus V_2 \oplus \cdots \oplus V_m = (V_1 \oplus V_2 \oplus \cdots \oplus V_{m-1}) \oplus V_m.$$

Example 1.3.1. Let's look at the Kronecker quiver again $1 \rightrightarrows 2$. The direct sum of two representations is depicted below.

$$V : k \overset{1}{\underset{0}{\rightrightarrows}} k$$

$$\oplus$$

$$W : k \overset{\begin{bmatrix}1\\0\end{bmatrix}}{\underset{\begin{bmatrix}1\\0\end{bmatrix}}{\rightrightarrows}} k^2$$

$$\|$$

$$V \oplus W : k \oplus k \overset{\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}}{\underset{\begin{bmatrix}0&0\\0&1\\0&0\end{bmatrix}}{\rightrightarrows}} k \oplus k^2 \cong k^2 \overset{\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}}{\underset{\begin{bmatrix}0&0\\0&1\\0&0\end{bmatrix}}{\rightrightarrows}} k^3$$

Exercise 1.3.2. Construct the isomorphism in Example 1.3.1.

Now given two representations, we know how to build a new one by taking the direct sum; but what about the other way around? If we are given a representation, can we always decompose it into a direct sum? The answer to this is no:

Definition 1.3.2. A representation $V \neq 0$ of a quiver Q is called **indecomposable** if whenever $V \cong W \oplus X$, then either $W = 0$ or $X = 0$. If V is not indecomposable, we call it **decomposable**.

Example 1.3.2.

1. In Example 1.3.1, V is indecomposable, but W is not. It decomposes into two indecomposables:

$$W : k \overset{\begin{bmatrix}1\\0\end{bmatrix}}{\underset{\begin{bmatrix}1\\0\end{bmatrix}}{\rightrightarrows}} k^2 \cong k \overset{1}{\underset{1}{\rightrightarrows}} k \oplus 0 \overset{0}{\underset{0}{\rightrightarrows}} k$$

2. In Example 1.2.1, V is decomposable.

3. In Example 1.2.2, both V and W are indecomposable.

4. In Example 1.2.6, both V and W are indecomposable.

At this point, we may ask if *any* decomposable representation can decompose into the sum of *indecomposable* representations. The answer to this is yes, and more!

Theorem 1.3.1. *[Krull-Schmidt] Let Q be a quiver and V a representation. Then $V \cong V_1 \oplus V_2 \oplus \cdots \oplus V_t$ where for each i, $V_i \in repQ$ is unique up to reordering.*

By unique we usually mean up to isomorphism in this context and to prove existence we could use an inductive argument as in Schiffler's book. For a proof of uniqueness, see the references in Schiffler's book.

If we think about what this theorem is saying, this idea isn't entirely new! It says that any object in rep$Q$ is 'built out of' indecomposables. We could in some sense think of these indecomposables as basis elements and any representation as a linear combination of indecomposables where all the coefficients are nonnegative integers and the addition is taking direct sums. More importantly, to study rep$Q$, it suffices to only look at the indecomposables!

Goal: Generally speaking, the goal of representation theory is to classify all isoclasses of indecomposable representations and the morphisms between them.

One thing to notice is that this idea of objects breaking into indecomposable parts does not always hold!

Zooming Out 3. *A 'sufficiently nice' category $\mathscr{C}$ in which every object decomposes into a finite direct sum of 'nice' objects is called a* ***Krull-Schmidt category***

Okay, back to building new representations from old ones. Remember the notion of subspaces from linear algebra?

Definition 1.3.3. Let L, V, and W be three objects in rep$Q$ for some Q.

- We call L a **subrepresentation** of V if there exists an injective morphism $i : L \hookrightarrow V$ (the hooked arrow signifies that the map is an injection).
- Two subrepresentations of V, $i_1 : L \hookrightarrow V$ and $i_2 : W \hookrightarrow V$, are **equal** if and only if $i_1(L) = i_2(W)$.

An analogous definition is the following: $L = (L_i, \phi_\alpha|_{L_i})$ is a **subrepresentation** of $V = (V_i, \phi_\alpha)$ if L_i is a subvector space of V_i for each i and $\phi_\alpha(L_i) \subset L_j$ whenever α is an arrow from i to j. The notation $\phi_\alpha|_{L_i}$ signifies the restriction of ϕ_α to the subspace L_i.

Wait! Why are the two definitions the same?

Example 1.3.3. Let Q be the quiver $1 \to 2$. Then the top representation is a subrepresentation of the bottom representation:

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[d, "0"'] & \Bbbk \arrow[d, "1"] \\ \Bbbk \arrow[r, "1"] & \Bbbk \end{tikzcd}$$

What this definition is saying is that to compare subrepresentations, it doesn't suffice to look at just the representations. We need to look at how the representations are embedded into the parent (bigger) representation. We can use this idea of subrepresentations to build new representations out of morphisms! Recall from linear algebra that given a linear transformation between two vector spaces $f : V \to W$, we could study all the elements $v \in V$ such that $f(v) = 0$; ie, the elements that f sends to zero. We called this the kernel of f, denoted by $\ker f$, which was equivalent to the nullspace of the standard matrix associated to f. This nullspace was a sub-vector space of the domain of f, which is V. In a similar way, given a morphism between two representations of Q, we want to define the kernel of the morphism, which ends up being a subrepresentation of the domain.

Definition 1.3.4. Let $V = (V_i, \phi_\alpha)$ and $W = (W_i, \psi_\alpha)$ be two representations of a quiver Q and $f : V \to W$ be a a morphism from V to W. Then for each i, $f_i : V_i \to W_i$ is a linear transformation between vector spaces, so it has a kernel. Define $L_i = \ker f_i$. Further, for each $\alpha \in Q_1$, define $\xi_\alpha(x) = \phi_\alpha(x)$ for all $x \in L_i$. We call ξ_α the **restriction** of ϕ_α to the subspace L_i. We call $L = (L_i, \xi_\alpha)$ the **kernel** of the morphism f.

This seems like the 'right' definition, but to be sure, we need to verify that L is actually a representation of Q.

Lemma 1.3.2. *The kernel of $f : V \to W$, denoted by $L = (L_i, \xi_\alpha)$, is an object of rep$Q$.*

Proof. To be a finite-dimensional representation of Q, we need that L_i is a finite-dimensional vector space for each i, which we have, and also that $\xi_\alpha : L_i \to L_j$ is a linear transformation from L_i to L_j. We know that ξ_α is linear because ϕ_α is and ξ_α is just the restriction of ϕ_α. What we don't know is that ξ_α sends stuff in the kernel of f_i to stuff in the kernel of f_j; that is, we don't know that $\xi_\alpha(L_i) \subset L_j$. Again, since ξ_α is just the restriction of ϕ_α, this is equivalent to asking whether ϕ_α sends stuff in the kernel of f_i to stuff in the kernel of f_j. But remember that f is a morphism of representations! So we have the commutivity relation $\psi_\alpha(f_i(x)) = f_j(\phi_\alpha(x))$. Here's a picture to keep in mind:

$$\begin{tikzcd} L_i \arrow[r, "\xi_\alpha = \phi_\alpha"] \arrow[d, "f_i"'] & L_j \arrow[d, "f_j"] \\ W_i \arrow[r, "\psi_\alpha"] & W_j \end{tikzcd}$$

If $x \in \ker f_i$, then $f_i(x) = 0$ by definition and $\psi_\alpha(f_i(x)) = \psi_\alpha(0) = 0$ since ψ_α is a linear transformation. By commutivity, we have that $0 = f_j(\phi_\alpha(x))$, which means that $\phi_\alpha(x) \in \ker f_j$ by definition. This allows us to conclude that if $x \in \ker f_i$, then $\phi_\alpha(x) \in \ker f_j$. Therefore ξ_α sends stuff in the kernel of f_i to stuff in the kernel of f_j and L is a well-defined finite-dimensional representation of Q. $\square$

Example 1.3.4. Recall the morphism $f = (1, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, 0)$ from Example 1.2.3. Here it is again:

$$\begin{tikzcd}
V : \Bbbk \arrow[r, "\begin{bmatrix}1\\0\\0\end{bmatrix}"] \arrow[d, "1"'] & \Bbbk^3 \arrow[r, "\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}"] \arrow[d, "\begin{bmatrix}1&0&0\end{bmatrix}"] & \Bbbk^2 \arrow[d, "0"] \\
W : \Bbbk \arrow[r, "1"] & \Bbbk \arrow[r, "0"] & 0
\end{tikzcd}$$

Let's find the kernel of f by starting with the vector spaces. We have that $f_1 = 1$ is the identity map. Thus it is an injective linear transformation between two finite dimensional vector spaces, so it has a trivial kernel (the null space of the corresponding matrix is only the zero vector). Thus $L_1 = 0$. Now to find the kernel of f_2, we need to compute the nullspace of the matrix $\begin{bmatrix}1 & 0 & 0\end{bmatrix}$; which consists of vectors $\begin{bmatrix}x\\y\\z\end{bmatrix} \in \Bbbk^3$ such that

$$0 = \begin{bmatrix}1 & 0 & 0\end{bmatrix} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = x.$$

From this we conclude $x = 0$, and so the kernel of f_2 has basis

$$\left\{ \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}.$$

Therefore the kernel of f_2 is a two dimensional subspace of $\Bbbk^3$, so $L_2 \cong \Bbbk^2$. Finally, $L_3 = \ker f_3$. Since f_3 is the zero map, it sends everything to 0, so $L_3 = \Bbbk^2$.

Now we need to find the maps: $\xi_\alpha = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\big|_0$. Any map restricted to the zero vector space is just the zero map, so $\xi_\alpha = 0$. On the other hand, ξ_β is more interesting. We need to restrict

$$\phi_\beta = \begin{bmatrix}1 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

to L_2, which is a two dimensional subspace of $\Bbbk^3$. But here it is important to remember *which* two dimensional subspace L_2 is! It is the bottom two components of the vector $\begin{bmatrix}x\\y\\z\end{bmatrix}$. To restrict ϕ_β to the last to components, we need to keep the last two columns of the matrix and discard the first column (why?). Therefore $\xi_\beta = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ and $\ker f = L$ is isomorphic to the following representation:

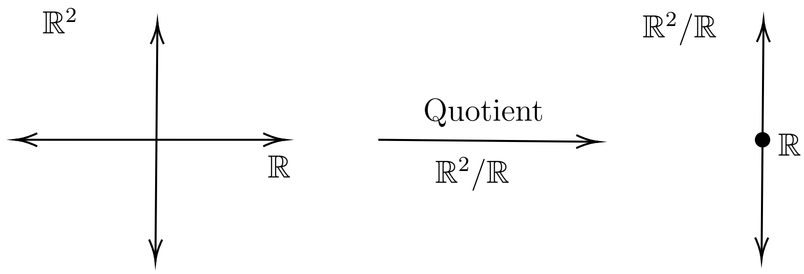
$$\begin{tikzcd}
L : 0 \arrow[r, "0"] & \Bbbk^2 \arrow[r, "\begin{bmatrix}1&0\\0&1\end{bmatrix}"] & \Bbbk^2
\end{tikzcd}$$

Wait! Is L indecomposable?

1.4 Quotients and Cokernels

Question 1.4. How else can we use morphisms to produce new representations?

Before we answer this question, we need the notion of a quotient vector space. Intuitively, a quotient of a vector space V by a subspace W, denoted V/W, is like taking V and identifying all vectors in W as one vector. Although the reader may not see this after reading the formal definition, we can think of quotient spaces as 'gluing' all the vectors in W together to make one vector. One purpose of the quotient space is to study all the vectors in V that are *not* in W, as we will see at the end of this section.



Definition 1.4.1. Let V be a $\Bbbk$-vector space and $W \subset V$ be a subspace. Then the **quotient of** V **by** W, denoted by V/W, is the collection of sets of the form $[v] = v + W = \{v + w : w \in W\}$. These sets are called **left cosets of** W **in** V and they are equivalent to the analogously defined **right cosets of** W **in** V because... why?

There are two analogies that coincide more with the definition, namely division of integers and subtraction of exponents. For instance, when we divide two integers, we look at the remainder. Sometimes, the remainder gives us the intuition behind the number of 'different' cosets in a quotient space. This mainly shows up in the context of 'group theory' and quotient groups like $\mathbb{Z}/n\mathbb{Z}$, whatever those are. On the other hand, when we divide polynomials like x^5/x^3, we subtract the exponents! In the cases that we will see, this is the intuition behind the number of 'different' cosets. But there are infinitely many cosets since there are infinitely many vectors $v \in V$! So what do we mean by 'different'?

Lemma 1.4.1. *Let V be a $\Bbbk$-vector space and $W \subset V$ be a subspace. Then the quotient V/W is a $\Bbbk$-vector space.*

Proof. To show that V/W is a vector space, we need to define addition, produce a 0 vector, define scalar multiplication, and show that these operations satisfy the axioms of a $\Bbbk$-vector space, so let's go.

Let $v_1 + W$ and $v_2 + W$ be two cosets in V/W. Define **addition** in the quotient space by $v_1 + W + v_2 + W = v_1 + v_2 + W$. Now we need to check that this is well defined. First we have $v_1 + v_2 + W \in V/W$ since V is a vector space and $v_1 + v_2 \in V$. Now we need to show that addition is independent of the element we pick from each coset. Pick an element from the coset $v_1 + W$, say $v_1 + w_1$, and another element $v_2 + w_2$ from the coset $v_2 + W$ where both w_1 and w_2 are in the subspace W. Then adding these normally we get $v_1 + w_1 + v_2 + w_2 = v_1 + v_2 + w_1 + w_2$. Since W is a subspace, $w_1 + w_2 \in W$, so $v_1 + v_2 + w_1 + w_2 \in v_1 + v_2 + W$. Since w_1 and w_2

were arbitrary, we have shown that this holds for any elements of $v_1 + W$ and $v_2 + W$, so addition is independent of choice of representative and is therefore well-defined. In other words, since W is closed under addition, it does not matter which elements of W we pick!

Now, let $0 \in V$ be the zero vector. Then $[0] = 0 + W = \{0 + w = w \in W\} = W$ is a candidate for the zero element in V/W. Actually, $W = w + W$ for *any* $w \in W$ (why?), including zero! So therefore $[w] = [0]$, or analogously, $w + W = 0 + W = W$ for any $w \in W$. Conversely, if $[v] = [0]$, then $\{v + w : w \in W\} = \{w \in W\}$. In other words, every element of $\{v + w : w \in W\}$ is an element of W. Since $0 \in W$, this implies that $v + 0 \in W$, so $v \in W$. Therefore we conclude $[v] = [0]$ if and only if $v \in W$. Here we have taken all of W and turned it into *one* element, namely the zero vector!

Let $c \in \Bbbk$ and define **scalar multiplication** as $c(v_1 + W) = cv_1 + W$. Note that $cv_1 + W \in V/W$, so to finish checking that this is well defined, let $v_1 + w \in v_1 + W$. Then $c(v_1 + w) = cv_1 + cw$. Since W is a vector space, $cw \in W$. Therefore $c(v_1 + w) \in cv_1 + W$. Since w was arbitrary, this holds for every element in $v_1 + W$ and thus scalar multiplication is well-defined.

Exercise 1.4.1. Show that these operations satisfy the remaining axioms of a $\Bbbk$-vector space.

□

Lemma 1.4.2. *Let $[v_1], [v_2] \in V/W$. Then $[v_1] = [v_2]$ if and only if $v_1 - v_2 \in W$.*

Proof. Suppose $[v_1] = [v_2]$. Then since V/W is a vector space, $[v_1] - [v_2] = [0]$. By the definition of addition in the quotient space, $[v_1]-[v_2] = v_1+W-v_2+W = v_1-v_2+W = [v_1-v_2]$. Therefore $[0] = [v_1] - [v_2] = [v_1 - v_2]$, which implies that $v_1 - v_2 \in W$. Conversely, suppose $v_1 - v_2 \in W$. Then by the work done in the proof of the previous lemma, $[0] = [v_1 - v_2] = [v_1] - [v_2]$. Adding $[v_2]$ to both sides gives $[v_2] = [v_1]$ as desired. □

Since the quotient of a vector space by a subspace is a vector space, it has a basis! By 'different' cosets, we mean linearly independent ones!

Example 1.4.1. Consider $V = \mathbb{R}^2$ and the subspace $W = \operatorname{span}(\{(1, 0)\})$. Then we can take the quotient: $\mathbb{R}^2/\operatorname{span}(\{(1, 0)\}) = \{v+\operatorname{span}(\{(1, 0)\})\}$. Let's find a basis for this vector space. Let $[v] \in V/W$. Then either $v \in W$ or not. If it is, then $[v] = [0]$ by the previous lemma, so let's suppose $v \notin W$, hence v is a vector in $\mathbb{R}^2$ that is *not* in the span of the vector $(1, 0)$. This allows us to conclude that v must be of the form (a, b) where b is *nonzero*. The vector v can be written as a linear combination $v = a(1, 0) + b(0, 1) \in \{(0, b) + \operatorname{span}\{(1, 0)\}\} = [(0, b)]$. Since $[v]$ was arbitrary, we conclude that $V/W \subset \operatorname{span}(\{(0, 1) + W\})$. Since $\{(0, 1) + W\}$ spans V/W and only has one element, which makes it minimal, it is a basis for V/W. We conclude that V/W is a one-dimensional real vector space, so $\mathbb{R}^2/\operatorname{span}(\{(1, 0)\}) \cong \mathbb{R}$.

Example 1.4.2. Let $V = \mathbb{R}^5$ and $W = \operatorname{span}(\{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)\}) \cong \mathbb{R}^3$. Then the quotient space $V/W \cong \mathbb{R}^2$ has basis $\{[(0, 0, 0, 1, 0)], [(0, 0, 0, 0, 1)]\}$.

Exercise 1.4.2. Verify the statements in the previous example.

Remember when we first started learning about exponents? We said that 'when we divide, we subtract the exponents'. Well that is what's going on here in terms of dimension! In the previous example, the subtraction of 5-3 = 2 gives us the number of distinct cosets.

Proposition 1.4.3. *Let V be an n-dimensional $\mathbb{k}$-vector space and W an m-dimensional subspace with $m \leq n$. Then $V/W \cong \mathbb{k}^{n-m}$ is an $n-m$-dimensional $\mathbb{k}$-vector space.*

Exercise 1.4.3. Prove the previous proposition!

Okay, so now that we have the idea of a quotient vector space, let's turn to that of a quotient representation. Let Q be a quiver, $M = (M_i, \phi_\alpha)$ a $\mathbb{k}$-representation of Q, and $N = (N_i, \phi_\alpha)$ a subrepresentation of M.

Definition 1.4.2. We define the **quotient** of M by N to be the representation $M/N = (M_i/N_i, \chi_\alpha)$ where χ_α is the map between the quotient vector spaces induced by ϕ_α, namely, $\chi_\alpha(m_i + N_i) = \phi_\alpha(m_i) + N_j$.

Notice, we need to show that this map is well defined! This means that the map is independent of the representative we choose from each coset and that the map sends equal cosets to equal cosets. First, take two elements from the same coset $m_i + N_i$, say $m_i + n_i$ and $m_i + n_i'$. Then since N is a subrepresentation, ϕ_α sends N_i to N_j, so we get that $\chi_\alpha(m_i + n_i + N_i) = \phi_\alpha(m_i) + \phi_\alpha(n_i) + N_j$ by definition. Because $\phi_\alpha(n_i) \in N_j$, we get that $\chi_\alpha(m_i + n_i + N_i) = \phi_\alpha(m_i) + \phi_\alpha(n_i) + N_j = \phi_\alpha(m_i) + N_j$ (why?). Using the same argument, $\chi_\alpha(m_i + n_i' + N_i) = \phi_\alpha(m_i) + \phi_\alpha(n_i') + N_j = \phi_\alpha(m_i) + N_j$. Therefore, the map χ_α is independent of the choice of representative!

Now we must show that if we have two equal cosets, say $m + N_i = m' + N_i$, that $\chi_\alpha(m + N_i) = \chi_\alpha(m' + N_i)$. Since we assume that these two cosets are equal, by Lemma 1.4.2, we know that $m - m' \in N_i$. We have $\chi_\alpha(m + N_i - m' + N_i) = \chi_\alpha(m - m' + N_i) = \phi_\alpha(m - m') + N_j = \phi_\alpha(m) - \phi_\alpha(m') + N_j$. We know that $\phi_\alpha(m - m') \in N_j$ since N is a subrepresentation of M. Moreover, since ϕ_α is linear, we conclude that $\phi_\alpha(m - m') = \phi_\alpha(m) - \phi_\alpha(m') \in N_j \implies \phi_\alpha(m) + N_j = \phi_\alpha(m') + N_j$ again by Lemma 1.4.2, allowing us to conclude that χ_α is well-defined.

An alternative definition is that the representation N is a **quotient** of M if there exists a surjective morphism $M \twoheadrightarrow N$. We will see why these two definitions are the same shortly (it follows from the first isomorphism theorem!).

Example 1.4.3. Let Q be the quiver $1 \to 2$. Then the bottom representation is a quotient of the top representation:

$$\begin{tikzcd}
\mathbb{k} \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{k} \arrow[d, "0"] \\
\mathbb{k} \arrow[r, "0"] & 0
\end{tikzcd}$$

We are now ready to obtain a new representation from a morphism! Remember from linear algebra when we have a linear transformation between two vector spaces, say $f : V \to W$, then the image of f, or the column space of the standard matrix of f, is a sub-vector space of W. Therefore, we can take the quotient of W by the image of f to get $W/\mathrm{im}f$, which we call the **cokernel** of the linear transformation f. In a similar way, now let $f : M \to N$ be a morphism between two representations $M = (M_i, \phi_\alpha)$ and $N = (N_i, \psi_\alpha)$ of some quiver Q. Then the **image** of f is the representation $\mathrm{im}f = (\mathrm{im}f_i = f_i(M_i), \psi_\alpha)$, which is by definition a subrepresentation of N. Using this, we have the following definition.

Definition 1.4.3. Let $f : M \to N$ be a morphism between two representations $M = (M_i, \phi_\alpha)$ and $N = (N_i, \psi_\alpha)$ of some quiver Q. The **cokernel** of f is the representation given by $N/\mathrm{im}f$.

Let's unpack this definition a bit. The cokernel of the morphism f is $N/\mathrm{im}f = (N_i/f_i(M_i), \chi_\alpha)$ where χ_α is the map induced on the quotient by ψ_α, that is, $\chi_\alpha(n_i + f_i(M_i)) = \psi_\alpha(n_i) + f_j(M_j)$ for each arrow $\alpha : i \to j$. Let's see an example.

Example 1.4.4. Define $f = (0, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix})$ as shown in blue below.

$$\begin{tikzcd}
M: 0 \arrow[r, "0"] \arrow[d, "0"'] & \Bbbk \arrow[r, "1"] \arrow[d, "\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}"] & \Bbbk \arrow[d, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"] \\
N: \Bbbk \arrow[r, "\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}"'] & \Bbbk^3 \arrow[r, "\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}"'] & \Bbbk^2
\end{tikzcd}$$

Let's compute $\mathrm{im}f_i$ for each i. For $i = 1$, we have that $f_1 = 0$, so $\mathrm{im}f_i = 0$. For $i = 2$,

$$f_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

so $\mathrm{im}f_2$ is the subspace of $\Bbbk^3$ whose basis is

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Similarly, $\mathrm{im}f_3$ is the subspace of $\Bbbk^2$ whose basis is

$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

Now we compute the vector spaces of the cokernel of f. At vertex 1, we have the vector space $\Bbbk/0 = \Bbbk$. At vertex 2, we have $\Bbbk^3/\mathrm{im}f_2$, which is the vector space with basis

$$\left\{ \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mathrm{im}f_2 \right\}, \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \mathrm{im}f_2 \right\} \right\} \cong \Bbbk^2;$$

call this vector space V. Finally at vertex 3, we have $\Bbbk^2/\text{im} f_3$, which is the vector space with basis

$$\left\{\left\{\begin{bmatrix}1\\0\end{bmatrix} + \text{im} f_3\right\}\right\} \cong \Bbbk;$$

call this vector space W. Therefore, the cokernel of f is given by the following representation:

$$\Bbbk \xrightarrow{\begin{bmatrix}1\\0\\0\end{bmatrix} + V} \Bbbk^2 \xrightarrow{\begin{bmatrix}1 & 1\end{bmatrix} + W} \Bbbk \qquad \cong \qquad \Bbbk \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} \Bbbk^2 \xrightarrow{\begin{bmatrix}1 & 1\end{bmatrix}} \Bbbk$$

$$x \longmapsto \begin{bmatrix}x\\0\\0\end{bmatrix} + V \qquad \begin{bmatrix}x\\y\\0\end{bmatrix} + V \longmapsto x + y + W$$

We're going to finish this section with one of my favorite theorems; let's go back to vector spaces for a few moments. Given a linear transformation $f : V \to W$ between two finite dimensional vector spaces, then f splits the vector space V into two parts. Let $v \in V$ be a vector. Then either f sends v to zero, or it sends v to something in W that isn't 0. So in a way, the linear transformation splits the vector space V into the kernel of f (the stuff that gets sent to zero), and the stuff that does not get sent to zero! Because the collection of vectors in V that are not in the kernel of f does not form a subspace of V (why?), we can realize the vectors that don't get sent to zero as $V/\text{ker} f$; that is, take V, glue together all the vectors that f sends to zero, and make them zero.

But quotient spaces are somewhat more difficult to think about than normal vector spaces because of cosets, so we really want to realize $V/\text{ker} f$ as a regular vector space. Let's look at $\text{im} f \subset W$. The nontrivial vectors in the subspace $\text{im} f$ are precisely the vectors in V that don't get sent to zero and the trivial vector in $\text{im} f$ is precisely all the vectors in the kernel of f! So it seems like there should be some relation between the image of f and the quotient $f/\text{ker} f$. This is precisely the statement of one of my favorite theorems! It was proven in 1927 by Emmy Noether for representations of quivers (really 'modules' which are just spicy vector spaces) and some special cases were proven before this by Richard Dedekind. If the reader is unfamiliar with Emmy Noether, they should look her up because she lived a truly inspirational life in my opinion.

Theorem 1.4.4 (Noether's First Isomorphism Theorem)**.** *Let $f : V \to W$ be a linear transformation between vector spaces V and W. Then*

$$imf \cong V/kerf.$$

Proof. Consider the map $\varphi : V/\text{ker} \to \text{im} f$ given by $\varphi(v + \text{ker} f) = f(v)$. Then this is a linear transformation because f is. We first show it is well defined. Let $[v] = [v']$, so $v - v' \in \text{ker} f$ by Lemma 1.4.2. We have $\varphi([v] - [v']) = f(v - v') = f(v) - f(v')$ since f is linear. On the other hand, $v - v' \in \text{ker} f$, so $f(v - v') = 0 = f(v) - f(v')$. Thus $f(v) = f(v')$, so φ is well-defined. Let's show it is surjective: let $y \in \text{im} f$. Then by definition of $\text{im} f$, there exists some vector $x \in V$ such that $f(x) = y$. We have that $\varphi(x + \text{ker} f) = f(x) = y$, so φ is surjective. Finally to show injectivity, suppose that there are $[x], [x'] \in V/\text{ker} f$ such that $\varphi([x]) = \varphi([x'])$. This means $f(x) = f(x')$, so $f(x) - f(x') = 0$ and by the linearity of f, $f(x - x') = 0$. From this we conclude that $x - x' \in \ker f$, allowing us to conclude $[x] = [x']$ again by Lemma 1.4.2. Thus, φ is injective. Therefore, φ is a bijective linear transformation, hence an isomorphism, between $\operatorname{im} f$ and $V/\ker f$ as desired. $\square$

The statement for representations of quivers is the following:

Theorem 1.4.5 (Noether's First Isomorphism Theorem). *Let $f : M \to N$ be a morphism between two representations of Q called M and N. Then*

$$im f \cong M/ker f.$$

Exercise 1.4.4. Prove Theorem 1.4.5.

1.5 Exact Sequences and Abelian Categories

This section will be split into two parts that address the following questions:

Question 1.5. How can we further study representations using kernels and cokernels? Do kernels and cokernels always exist in every category?

We begin with studying kernels and cokernels in the realm of quiver representations. To do this, we first introduce possibly the most important notion in the study of representations of quivers:

Definition 1.5.1.

- A sequence of morphisms $L \xrightarrow{f} M \xrightarrow{g} N$ between three representations L, M, and N of Q is called **exact at** M if $\operatorname{im} f = \ker g$.
- A sequence of morphisms

$$\cdots \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to \dots$$

  is called **exact** if it is exact at M_i for each i.
- A **short exact sequence** is an exact sequence of the form

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

We can already prove something!

Lemma 1.5.1. *The sequence of representations $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is short exact if and only if f is injective, $im f = ker g$, and g is surjective.*

Proof. Suppose $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is short exact. Then by definition, $\mathrm{im}0 = \ker f$, so $\ker f = 0$ and f is injective. Similarly, since the sequence is exact, $\mathrm{im}g = \ker 0$. Since the zero map sends everything to 0, $\ker 0 = N$. Therefore $\mathrm{im}g = N$ and g is surjective. Finally it follows from the definition of exactness that $\mathrm{im}f = \ker g$.

Conversely, suppose f is injective, $\mathrm{im}f = \ker g$, and g is surjective. Then since f is injective, it has trivial kernel. Therefore $\ker f = 0 = \mathrm{im}0$, so the sequence is exact at L. Similarly, since g is surjective, $\mathrm{im}g = N = \ker 0$. Thus the sequence is exact at N. Finally we are given that the sequence is exact at M, so we have shown that the sequence is short exact and therefore proven the statement. □

Let's see how this is related to kernels and cokernels:

Example 1.5.1. Consider the morphism $f : M \to N$ in rep$Q$. Then we have an exact sequence (not short exact)

$$0 \to \ker f \xrightarrow{i} M \xrightarrow{f} N \xrightarrow{p} \mathrm{coker} f \to 0.$$

Recall from Definition 1.3.3, a subrepresentation is a representation along with an injective morphism. Here, $\ker f$ is a subrepresentation and i is the injective morphism. It is the inclusion morphism. Similarly, recall the analogous definition of a quotient representation from Definition 1.4.2. It was a representation equipped with a surjection onto it. Here, $\mathrm{coker} f$ is a quotient representation and p is the surjection onto it. We can also explicitly define p as follows. Recall that $\mathrm{coker} f = N/\mathrm{im} f$. Thus $p : N \to N/\mathrm{im} f$ is the map that sends $n \mapsto n + \mathrm{im} f = [n]$. This is called the **canonical quotient map**.

Wait! Why is this sequence exact?

Example 1.5.2. Let's shorten the exact sequence from the previous example. We have a short exact sequence given by

$$0 \to \ker f \xrightarrow{i} M \xrightarrow{\varphi} M/\ker f \cong \mathrm{im} f \to 0.$$

In this sequence, i is again the inclusion of the subrepresentation $\ker f$, and φ is the isomorphism the reader constructed in Exercise 1.4.4, which is analogous to φ from Theorem 1.4.4.

Wait! Why is this sequence short exact?

Okay, enough with the abstract examples. Let's see some concrete exact sequences.

Example 1.5.3. Let Q be the quiver $1 \to 2$ and consider the following exact sequence three representations written vertically as L, M, and N respectively.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "0"'] & 0 \arrow[d, "0"] \\
0 \arrow[r, "0"] \arrow[d, "0"'] & \Bbbk \arrow[d, "1"] \\
\Bbbk \arrow[r, "1"] \arrow[d, "1"'] & \Bbbk \arrow[d, "0"] \\
\Bbbk \arrow[r, "0"] \arrow[d, "0"'] & 0 \arrow[d, "0"] \\
0 \arrow[r, "0"] & 0
\end{tikzcd}$$

The morphisms written in blue are $f_0 = (0,0), f_1 = (0,1), f_2 = (1,0)$, and $f_3 = (0,0)$. The reader should check that $\mathrm{im} f_i = \ker f_{i+1}$ for each i. Since my fingers are tired from typing all that notation, now would be a good time to introduce a new shorthand notation for the representations L, M, and N. For the representation $M = \Bbbk \xrightarrow{1} \Bbbk$, we write $\begin{smallmatrix}1\\2\end{smallmatrix}$. There is one 1, which means there is a one dimensional vector space at vertex 1. There is one 2, which means there is a one dimensional vector space at vertex 2. The 1 is on top of the 2 because there is an *arrow* in the quiver equipped with a nonzero map (the identity map in this case). The notation for L is 2 and that for N is 1. We can't always use this notation, but in the case in which the isoclass of an indecomposable representation is determined by its dimension vector, we can. We can rewrite the short exact sequence using the shorthand notation:

$$0 \to 2 \xrightarrow{i} \begin{matrix}1\\2\end{matrix} \xrightarrow{p} 1 \to 0$$

where i is the inclusion f_1 and p is the projection f_2. Using this notation, roughly speaking, the subrepresentations are the ones given by the numbers on the bottom, and the quotient representations are given by the stuff on the top (why?). Another short exact sequence is

$$0 \to 2 \xrightarrow{f} 1 \oplus 2 \xrightarrow{g} 1 \to 0.$$

As we will soon see, this type of short exact sequence has a special name and moreover, we can always make short exact sequences like this.

Wait! What are the maps?

Example 1.5.4. Let Q be the quiver from Example 1.2.2, given by

$$\begin{tikzcd}
 & & 3 \arrow[dl, "\beta"'] \\
1 \arrow[r, "\alpha"] & 2 \arrow[dr, "\gamma"] & \\
 & & 4
\end{tikzcd}$$

and consider the representations $L = 2$, $M = \begin{smallmatrix}1 & & 3\\ & 2 & \end{smallmatrix}$, and $N = 1 \oplus 3$. The corresponding representations are

$$\begin{tikzcd} & & 0 \arrow[dl, "0"'] \\ L : 0 \arrow[r, "0"] & \Bbbk \arrow[dr, "0"] & \\ & & 0 \end{tikzcd} \qquad \begin{tikzcd} & & \Bbbk \arrow[dl, "1"'] \\ M : \Bbbk \arrow[r, "1"] & \Bbbk \arrow[dr, "0"] & \\ & & 0 \end{tikzcd}$$

$$\begin{tikzcd} & & \Bbbk \arrow[dl, "0"'] \\ N : \Bbbk \arrow[r, "0"] & 0 \arrow[dr, "0"] & \\ & & 0 \end{tikzcd}$$

We have a short exact sequence given by

$$0 \to 2 \to \begin{matrix} 1\,3 \\ 2 \end{matrix} \to 1 \oplus 3 \to 0.$$

Wait! What are the maps?

Example 1.5.5. Consider the Kronecker quiver $1 \rightrightarrows 2$. Then we have a short exact sequence given by

$$0 \to 2 \to \begin{matrix} 1 \\ 2\,2 \end{matrix} \to M \to 0$$

where M is the representation of Q with $\dim M = (1,1)$, the top arrow is the zero map, and the bottom arrow is the identity map.

Exercise 1.5.1. Write down the maps and indecomposable representations in Example 1.5.5.

To further study short exact sequences, we need some more definitions.

Definition 1.5.2.

- A morphism between representations $f : M \to N$ is called a **section** if it has a left inverse; that is, there exists a morphism $h : N \to M$ such that $h \circ f = 1_M$.
- A morphism between representations $f : M \to N$ is called a **retraction** if it has a right inverse; that is, there exists a morphism $h : N \to M$ such that $f \circ h = 1_N$.
- A short exact sequence
$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$
splits or is called **split exact**, if f is a section or g is a retraction.

We've already seen one!

Example 1.5.6. The final sequence in Example 1.5.3 splits. Define $h : 1 \oplus 2 \to 2$ by $h((x,y)) = y$. Then $h(f(x)) = h((0,x)) = x$, so $h \circ f = 1_2$.

It turns out, all of them are like this!

Proposition 1.5.2. *A short exact sequence of representations*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is split exact if and only if

$$M \cong L \oplus N.$$

The reader at this point is probably incredibly curious about how this is proved. A proof of this proposition can be found in Schiffler's book; but be aware, it is three pages long. Okay, time to zoom out. What if we aren't playing with representations but instead with vector spaces? Or groups, rings, topological spaces, smooth manifolds, or whatever category we want? Can we build short exact sequences? Well remember, the definition of a short exact sequence is dependent on the notion of kernels and cokernels! So to answer this question, we really need to answer the question, do kernels and cokernels exist in every category? But what the heck is a kernel in a category?

Zooming Out 4. *Let $\mathscr{C}$ be a category and $g : M \to N$ a morphism between two objects M and N. The* ***kernel*** *of g is an object L and a morphism $f : L \to M$ such that $g \circ f = 0$. This is good, this is what we want! This is how kernels behave in terms of representations of quivers! But wait... there's more! We also require that the kernel satisfies the following* ***universal property****, namely, for any object X and morphism $h : X \to M$ such that $g \circ h = 0$, it factors through L. That is, there a unique morphism $u : X \to L$ such that $f \circ g = h$. Hang in there, a picture might help.*

$$\begin{tikzcd} & X \arrow[dl, dotted, "u"'] \arrow[d, "h"] & \\ L \arrow[r, "f"] & M \arrow[r, "g"] & N \end{tikzcd}$$

This picture says, give us h and we can find u (which is why it's dotted) such that the triangle commutes. Yay! Commuting diagrams!

Okay so we have category rep$Q$, and a notion of a kernel of a morphism, Definition 1.3.4. Seems natural to check that our definition agrees with the categorical definition!

Lemma 1.5.3. *The kernel of a morphism of representations satisfies the universal property given in Zooming Out 4.*

Proof. Let $g : M \to N$ be a morphism in rep$Q$ and $L = \ker g \xhookrightarrow{f} M$ be the kernel of g where f is the inclusion. Then for each $i \in Q_0$, we have that $L_i = \ker g_i$, so g_i sends everything in L_i to zero. By construction then, we have $g_i \circ f_i = 0$. Since this holds for all $i \in Q_0$, we have that $g \circ f = 0$. It remains to show that $L \xhookrightarrow{f} M$ satisfies the universal property.

To do this, consider the morphism $X \xrightarrow{h} M$ such that $g \circ h = 0$. We need to construct a morphism $X \xrightarrow{u} L$ such that $f \circ u = h$. To construct this, we need to give names to the maps in our representations. Let $L = (L_i, \phi_\alpha)$, $M = (M_i, \psi_\alpha)$, and $X = (X_i, \xi_\alpha)$. Since $g \circ h = 0$, we have for any $x_i \in X_i$, $h(x_i) \in \ker g_i = L_i$. Thus $X_i \subset L_i$, and we can define $u_i(x_i) = h_i(x_i)$.

Since f is just the inclusion map, $f_i(u_i(x)) = u_i(x) = h_i(x)$, so $f \circ u = h$ since this holds for all i. The final thing we need to show is that u is in fact a morphism of representations (makes the boxes commute, $\phi_\alpha \circ u_i = \xi_\alpha \circ u_j$). But this follows from the fact that h is a morphism of representations and $u_i = h_i$ for each i! Since h is a morphism of representations, for any arrow $\alpha : i \to j$, $\psi_\alpha \circ h_i = \xi_\alpha \circ h_j$. Since ϕ_α is just the restriction of ψ_α, we have

$$\phi_\alpha \circ u_i = \psi_\alpha \circ h_i = \xi_\alpha \circ h_j = \xi_\alpha \circ u_j.$$

Since we have the commutivity relation for each α, we have that the kernel of f does satisfy the universal property and the lemma is proved. $\square$

Of course, we also have the *dual* statement for the *co*kernel:

Zooming Out 5. *Whenever we see words like* ***dual*** *or* ***co****, we should be thinking about 'flipping the arrows'. This idea will be formalized in the next section of these notes, but for now, let $\mathscr{C}$ be a category and $g : N \to M$ a morphism between two objects M and N. The* ***cokernel*** *of g is an object and morphism $M \xrightarrow{f} L$ such that $g \circ f = 0$ (just like for cokernels of representations) and f satisfies the following* ***universal property****.*

$$\begin{tikzcd} & X & \\ L \arrow[ur, dotted, "u"] & M \arrow[l, "f"] \arrow[u, "h"'] & N \arrow[l, "g"] \end{tikzcd}$$

For any morphism $h : M \to X$ such that $h \circ g = 0$, there exists a unique morphism $u : L \to X$ such that the triangle commutes: $u \circ f = h$.

Exercise 1.5.2. Show that the cokernel of a morphism in rep$Q$ satisfies the categorical definition in Zooming Out 5.

Now we finish the section by asking, do kernels and cokernels always exist in every category? The answer is no, but when they do we have a name for the category.

Zooming Out 6. *A category $\mathscr{C}$ is called an* ***abelian*** $\mathbb{k}$***-category*** *if the following hold.*

1. *$\mathscr{C}$ is a $\mathbb{k}$-category; ie $Hom_{\mathscr{C}}(M, N)$ is a $\mathbb{k}$-vector space for all $M, N \in Ob(\mathscr{C})$ and the composition of morphisms is bilinear ($(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$ and $f \circ (g_1 + g_2) = f \circ g_1 + f \circ g_2$).*

2. *$\mathscr{C}$ is additive; ie $\mathscr{C}$ has direct sums, and a zero object $0 \in Ob(\mathscr{C})$ such that the identity morphism $1_0 \in Hom_{\mathscr{C}}(0, 0)$ is the zero morphism.*

3. *Each morphism $f : M \to N$ has a kernel $i : K \to M$ and a cokernel $p : N \to C$ such that $coker(i) \cong ker(p)$.*

We certainly know an abelian category called repQ! Condition 1 is satisfied by Proposition 1.2.1. We know how to take direct sums of objects, Definition 1.3.1, and we know that the representation where all the vector spaces and all the morphisms are zero is the zero object, so we have 2. Finally we have kernels, Definition 1.3.4, and cokernels, Definition 1.4.3.

Let $f : M \to N$ be a morphism of representations with kernel $i : K \to M$ and cokernel $p : N \to C$. By Example 1.5.1, we have an exact sequence

$$0 \to K \xrightarrow{i} M \xrightarrow{f} N \xrightarrow{p} C \to 0.$$

Then we have

$$\begin{aligned} ker(p) &= im(f) && (\text{by exactness}) \\ &\cong M/ker(f) && (\text{by the Noether isomorphism theorem, Theorem 1.4.5}) \\ &= M/im(i) && (\text{by exactness}) \\ &= coker(i) && (\text{by definition}) \end{aligned}$$

Therefore $coker(i) \cong ker(p)$, 3 is satisfied, and repQ is an abelian $\Bbbk$-category.

1.6 (Hom) Functors

Question 1.6. Maps between objects in a category are called morphisms. Are there maps between categories?

The answer to this question is yes! They are called functors and we will spend this section defining them and exploring their properties. This section is pretty abstract, so throughout the reader should keep a particular abelian $\Bbbk$-category in mind, like rep$Q$.

Zooming Out 7. *Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A* ***(covariant) functor*** *F from $\mathscr{C}$ to $\mathscr{D}$ is a mapping that satisfies the following:*

1. *For each $X \in Ob(\mathscr{C})$, F assigns an object $F(X) \in Ob(\mathscr{D})$. (F sends objects to objects)*
2. *For each $f \in Hom_{\mathscr{C}}(X, Y)$, F assigns a morphism $F(f) : F(X) \to F(Y) \in Hom_{\mathscr{D}}(F(X), F(Y))$ (F sends morphisms to morphisms), such that:*
   - *$F(1_X) = 1_{F(X)}$ for all objects $X \in Ob(\mathscr{C})$. (F sends the identity to the identity)*
   - *$F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$, and $g : Y \to Z$ in $\mathscr{C}$. (F preserves composition of morphisms)*

Of course, this is pretty abstract stuff, so let's think of some examples.

Example 1.6.1. Let $\mathscr{C}$ be a category. Define the functor 1 from $\mathscr{C}$ to itself as follows.

1. On objects $X \in \mathrm{Ob}(\mathscr{C})$, $1(X) := X$.
2. On morphisms $f : X \to Y$, $1(f) = f$.

Notice by definition, this functor sends the identity to the identity and preserves composition of morphisms. This functor is called the **identity functor**.

Example 1.6.2. Recall from Example 1.2.7, quivers are categories! Consider the category given by the quiver $Q : 1 \xrightarrow{\alpha} 2$. Then we can define a functor F as follows:

1. On objects: $F(1) = 1$, and $F(2) = 2$.
2. On morphisms: $F(\alpha) = 0$, $F(e_1) = e_1$, and $F(e_2) = e_2$.

This functor preserves the identity by definition, and composition of morphisms (check!), so it is a well-defined functor from Q to $Q' : 1 \quad 2$.

Let's define a new functor G on Q by:

1. On objects: $G(1) = 1$, and $G(2) = 2$.
2. On morphisms: $G(\alpha) = \beta$, where $s(\beta) = 2$ and $t(\beta) = 1$, $G(e_1) = e_1$, and $G(e_2) = e_2$.

Then
$$G(e_2 \circ \alpha) = G(\alpha) = \beta = \beta \circ e_2 = G(\alpha) \circ G(e_2).$$
Also, we have
$$G(\alpha \circ e_1) = G(\alpha) = \beta = e_1 \circ \beta = G(e_1) \circ G(\alpha).$$
Therefore we don't have that G preserves composition of morphisms, but actually *reverses* composition (flips the arrows! See we said we would explain it)! The map G isn't a functor in the way we've defined it, which is why we call the functor defined in Zooming Out 7 a **covariant** functor because it preserves composition of morphisms. The assignment G *flips* all the compositions and is an example of a **contravariant** functor from Q to the quiver $\tilde{Q} : 1 \xleftarrow{\beta} 2$.

Zooming Out 8. *Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A* ***contravariant functor*** *F from $\mathscr{C}$ to $\mathscr{D}$ is a mapping that satisfies the following:*

1. *For each $X \in Ob(\mathscr{C})$, F assigns an object $F(X) \in Ob(\mathscr{D})$. (F sends objects to objects)*
2. *For each $f \in Hom_{\mathscr{C}}(X, Y)$, F assigns a morphism $F(f) : F(Y) \to F(X) \in Hom_{\mathscr{D}}(F(Y), F(X))$ (F reverses the morphisms (arrows)), such that:*
   - *$F(1_X) = 1_{F(X)}$ for all objects $X \in Ob(\mathscr{C})$. (F sends the identity to the identity)*
   - *$F(g \circ f) = F(f) \circ F(g)$ for all morphisms $f : X \to Y$, and $g : Y \to Z$ in $\mathscr{C}$. (F reverses composition of morphisms)*

We now turn our attention to a *very* important functor. Let $\mathscr{C}$ be a $\Bbbk$-category as in Zooming Out 6, so $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is a $\Bbbk$-vector space. Now lets think of Hom as a 'function' of sorts, kind of like a derivative. By fixing X, we can look at the morphisms *out of* X, denoted by $\mathrm{Hom}(X, -)$. We feed this assignment some object Y and what we get back is all the maps from X to Y, denoted by $\mathrm{Hom}(X, Y)$. And moreover, $\mathrm{Hom}(X, Y)$ is a $\Bbbk$-vector space! So $\mathrm{Hom}(X, -)$ eats objects in the category $\mathscr{C}$ and returns a $\Bbbk$-vector space, namely $\mathrm{Hom}(X, Y)$. We have thus defined an assignment between the objects of $\mathscr{C}$ and the objects in the category of $\Bbbk$-vector spaces, denoted by $\mathbf{Vect}_{\Bbbk}$. In order to make $\mathrm{Hom}(X, -)$ a functor, we need to extend this assignment to morphisms, whence let $f : Y \to Z$ be a morphism between the objects Y and Z in $\mathscr{C}$. We need to define $\mathrm{Hom}(X, -)(f) = \mathrm{Hom}(X, f) = f_*$, ie we need to use f to create a morphism between the vector spaces $\mathrm{Hom}(X, Y)$ and $\mathrm{Hom}(X, Z)$. A morphism between these two vector spaces would eat a morphism from X to Y and give us a morphism from X to Z. Let $\varphi \in \text{Hom}(X, Y)$. Then φ takes us from X to Y. After we do this, we could take f from Y to Z. The composition $f \circ \varphi$ is then a map from X to Z; that is $f \circ \varphi \in \text{Hom}(X, Z)$. We define $\text{Hom}(X, f)$ as the morphism from $\text{Hom}(X, Y) \xrightarrow{f_*} \text{Hom}(X, Z)$ via $\varphi \mapsto f \circ \varphi$. This is called the **pushforward** of φ because it pushes it forward through Y from being a map from X to Y, to a map from X to Z. A picture may help:

$$\begin{tikzcd}
X \arrow[d, "\varphi"'] \arrow[dr, "f\circ\varphi = f_*(\varphi) = \text{Hom}(X,f)(\varphi)"] & \\
Y \arrow[r, "f"] & Z
\end{tikzcd}$$

We call this the **covariant Hom functor**, which is a functor from any $\Bbbk$-category $\mathscr{C}$ to $\mathbf{Vect}_\Bbbk$. In particular, since rep$Q$ is abelian by Zooming Out 6, we can apply the covariant Hom functor to representations of quivers. *Dually* (we're going to be flipping arrows!!!), by looking at the maps *into* X, we can define the **contravariant Hom functor** $\text{Hom}(-, X)$ as follows. To objects Y of $\mathscr{C}$, we assign the $\Bbbk$-vector space $\text{Hom}(Y, X)$. To define morphisms, let $f \in \text{Hom}(Y, Z)$, and $\varphi \in \text{Hom}(Z, X)$. Then we define $\text{Hom}(-, X)(f) = \text{Hom}(f, X) = f^*$ as the **pullback** of φ; that is, $f^*(\varphi) = \varphi \circ f \in \text{Hom}(Y, X)$. Intuitively, we are pulling φ back from starting at Z to starting at Y:

$$\begin{tikzcd}
 & X \\
Y \arrow[ur, "\text{Hom}(f,X)(\varphi) = f^*(\varphi) = \varphi\circ f"] \arrow[r, "f"] & Z \arrow[u, "\varphi"']
\end{tikzcd}$$

Exercise 1.6.1. Show that the covariant and contravariant Hom functors are actually functors.

Okay, so given objects and morphisms in a $\Bbbk$-category $\mathscr{C}$, we have a way to get objects and morphisms in the category $\mathbf{Vect}_\Bbbk$. This means if we have a sequence of objects and morphisms in the category, we should get a sequence of objects and morphisms in $\mathbf{Vect}_\Bbbk$. If the category $\mathscr{C}$ also has kernels and cokernels, we can extend the definition of exact sequences, Definition 1.5.1, to any abelian $\Bbbk$-category (Do it!). It might be natural to wonder whether the Hom functor sends exact sequences in $\mathscr{C}$ to exact sequences in $\mathbf{Vect}_\Bbbk$.

Wait! Why do we have exact sequences in $\mathbf{Vect}_\Bbbk$? Is $\mathbf{Vect}_\Bbbk$ abelian as in Zooming Out 6?

Theorem 1.6.1. *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N$ be a sequence of objects and morphisms in an abelian $\Bbbk$-category $\mathscr{C}$. Then it is exact if and only if*

$$0 \to Hom(X, L) \xrightarrow{f_*} Hom(X, M) \xrightarrow{g_*} Hom(X, N)$$

is exact for every $X \in Ob(\mathscr{C})$.

Proof. This proof is quite challenging and may take a while to absorb. The reader should have a pencil in hand, jotting down their ideas while reading through this. It will probably take several reads to understand, so let's be patient and move through slowly!

We begin by proving the 'forward direction' which starts with assuming that

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N$$

is exact, so $\mathrm{ker} f = 0$ and $\mathrm{im} f = \mathrm{ker} g$. We wish to show that for any $X \in \mathrm{Ob}(\mathscr{C})$, the sequence

$$0 \to \mathrm{Hom}(X, L) \xrightarrow{f_*} \mathrm{Hom}(X, M) \xrightarrow{g_*} \mathrm{Hom}(X, N)$$

is exact. This means we must show: 1. $\mathrm{ker} f_* = 0$, 2. $\mathrm{im} f_* \subset \mathrm{ker} g_*$, and 3. $\mathrm{ker} g_* \subset \mathrm{im} f_*$.

1. To show $\mathrm{ker} f_* = 0$, we must select an arbitrary element in the kernel of f_* and show it is the zero morphism. Whence let $\varphi \in \mathrm{Hom}(X, L)$ be in the kernel of f_*; that is, $f \circ \varphi = 0$, and we will show that φ is the zero morphism. Thus for any $x \in X$, $f(\varphi(x)) = 0$, which means that $\varphi(x) \in \mathrm{ker} f$ for all x by definition. But since we are assuming that $\mathrm{ker} f = 0$, this means that $\varphi(x) = 0$ for all x, which allows us to conclude that $\varphi = 0$. Therefore $\mathrm{ker} f_* = 0$.

2. To show $\mathrm{im} f_* \subset \mathrm{ker} g_*$, we must select an arbitrary element in $\mathrm{im} f_*$ and show that it also lives in $\mathrm{ker} g_*$. So let $\varphi \in \mathrm{Hom}(X, M)$ be in the image of f_*. Then by definition of image, there is some morphism $\psi \in \mathrm{Hom}(X, L)$ that f_* maps to φ; that is, $f \circ \psi = \varphi$. This means that for all $x \in X$, $f(\psi(x)) = \varphi(x)$, so $\varphi(x) \in \mathrm{im} f$ for all x by definition. But we're assuming that $\mathrm{im} f = \mathrm{ker} g$! So that means that for all x, $\varphi(x) \in \mathrm{ker} g$ which means that $f(\psi(x)) \in \mathrm{ker} g$ for all x since $f(\psi(x)) = \varphi(x)$. Therefore, since this holds for all x, $\varphi = f \circ \psi \in \mathrm{ker} g$, which shows that $\mathrm{im} f_* \subset \mathrm{ker} g_*$.

3. To show $\mathrm{ker} g_* \subset \mathrm{im} f_*$, we must select an arbitrary element in $\mathrm{ker} g_*$ and show it also lives in $\mathrm{im} f_*$. So let $\varphi \in \mathrm{Hom}(X, M)$ live in $\mathrm{ker} g_*$, which means that $g \circ \varphi = 0$ by definition. But recall the universal property of the kernel discussed in Zooming Out 4! Since $g \circ \varphi = 0$, it must factor through the kernel of g, which we assume to be equal to the image of f:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{im} f = \mathrm{ker} g \arrow[r, "i"] & M \arrow[r, "g"] & N \\
& & X \arrow[u, "\varphi"] \arrow[ul, dotted, "\psi"] \arrow[ur, "0=g\circ\varphi"'] &
\end{tikzcd}$$

In this diagram, the existence of the map ψ such that $i \circ \psi = \varphi$ follows from the universal property of the kernel where i is the inclusion of the kernel of g into M. So for all $x \in X$, $\psi(x) \in \mathrm{im} f$, which allows us to conclude that $i \circ \psi \in \mathrm{im} f_*$. Since $i \circ \psi = \varphi$, we have shown that $\varphi \in \mathrm{im} f_*$.

Points 1., 2., and 3. combined show that $\ker f_* = 0$ and $\operatorname{im} f_* = \ker g_*$, establishing the exactness of the sequence

$$0 \to \operatorname{Hom}(X, L) \xrightarrow{f_*} \operatorname{Hom}(X, M) \xrightarrow{g_*} \operatorname{Hom}(X, N).$$

Conversely, let's prove the 'reverse direction' and suppose that

$$0 \to \operatorname{Hom}(X, L) \xrightarrow{f_*} \operatorname{Hom}(X, M) \xrightarrow{g_*} \operatorname{Hom}(X, N)$$

is exact for all $X \in \operatorname{Ob}(\mathscr{C})$, so $\ker f_* = 0$ and $\operatorname{im} f_* = \ker g_*$. We wish to now show that

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N$$

is exact. This requires showing: 1. $\ker f = 0$, 2. $\operatorname{im} f \subset \ker g$, and 3. $\ker g \subset \operatorname{im} f$. To do this, we use the fact that the Hom sequence is exact *for any choice of* X. This allows us to pick particularly nice choices of X to prove things we want.

1. To show $\ker f = 0$, we pick $X = \ker f$. Then we are assuming that

$$0 \to \operatorname{Hom}(\ker f, L) \xrightarrow{f_*} \operatorname{Hom}(\ker f, M) \xrightarrow{g_*} \operatorname{Hom}(\ker f, N)$$

   is an exact sequence. Let $i : \ker f \to L$ be the inclusion of the kernel of f into L. Then by definition of kernel, $f_*(i) = f \circ i = 0$, which implies that $i \in \ker f_*$ by definition. But $\ker f_* = 0$ by our assumption! Thus $i = 0$; that is, the inclusion of the kernel of f into L is the zero map. This happens if and only if $\ker f$ is the zero object, so $\ker f = 0$.

2. To show $\operatorname{im} f \subset \ker g$, we will take $X = L$. Then we are assuming the exactness of the sequence:

$$0 \to \operatorname{Hom}(L, L) \xrightarrow{f_*} \operatorname{Hom}(L, M) \xrightarrow{g_*} \operatorname{Hom}(L, N).$$

   In particular, the identity morphism 1_L lives in $\operatorname{Hom}(L, L)$, so $f_*(1_L) = f \circ 1_L \in \operatorname{im} f_*$. But by exactness, $\operatorname{im} f_* = \ker g_*$, so $f \circ 1_L \in \ker g_*$. This means that for every $l \in L$, $f(1_L(l)) = f(l) \in \ker g$. Since this holds for all l, we conclude that $\operatorname{im} f = f(L) \in \ker g$, so $\operatorname{im} f \subset \ker g$.

3. Finally, we must show $\ker g \subset \operatorname{im} f$. Take $X = \ker g$ and let $i : \ker g \to M$ be the inclusion. Then we assume the exactness of the sequence:

$$0 \to \operatorname{Hom}(\ker g, L) \xrightarrow{f_*} \operatorname{Hom}(\ker g, M) \xrightarrow{g_*} \operatorname{Hom}(\ker g, N).$$

   Notice, $g \circ i = 0$ by construction, so $i \in \ker g_*$. Since we assume that $\ker g_* = \operatorname{im} f_*$, we have that $i \in \operatorname{im} f_*$. Thus by definition of image, there is some morphism $\varphi : \ker g \to L$ such that $f \circ \varphi = i$. This means that for all $m \in \ker g$, we have $f(\varphi(m)) = i(m)$, so $\ker g \subset \operatorname{im} f$.

Points 1., 2., and 3. combined show that $\ker f = 0$ and $\operatorname{im} f = \ker g$, establishing the exactness of the sequence

$$0 \to L \xrightarrow{f} M \xrightarrow{g_*} N.$$

$\square$

We will now present a corollary to this theorem, for which we omit the proof. A proof of this corollary can be found in Schiffler's book.

Corollary 1.6.2. *In an abelian $\Bbbk$-category $\mathscr{C}$, a sequence*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

splits if and only if for every $X \in Ob(\mathscr{C})$, the following sequence is exact:

$$0 \to Hom(X, L) \xrightarrow{f_*} Hom(X, M) \xrightarrow{g_*} Hom(X, N) \to 0.$$

Moreover, if

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

splits then so does

$$0 \to Hom(X, L) \xrightarrow{f_*} Hom(X, M) \xrightarrow{g_*} Hom(X, N) \to 0.$$

At this point, the reader may be wondering what this has to do with quivers. Well first, this abstract nonsense and diagram chasing is inherently interesting to some (the author). Second, what we have just done is quite powerful. We've proven things for *any* abelian $\Bbbk$-category! So any collection of objects and morphisms that satisfy Zooming Out 6 will also satisfy these theorems! This means that we've proven these theorems for short exact sequences of representations of quivers, $\Bbbk$-vector spaces, modules over rings, vector bundles over a fixed topological space, and sheaves over a topological space just to name a few! We should make a remark that if the exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ does *not* split, then g_* may not be surjective; that is, we may *not* get an exact sequence $0 \to \mathrm{Hom}(X, L) \xrightarrow{f_*} \mathrm{Hom}(X, M) \xrightarrow{g_*} \mathrm{Hom}(X, N) \to 0$. Here is an example:

Example 1.6.3. Recall Example 1.5.3. Q was the quiver $1 \to 2$ and we considered the representations 1, ${}^{1}_{2}$, and 2. They fit into a short exact sequence of the form

$$0 \to 2 \to \begin{matrix}1\\2\end{matrix} \to 1 \to 0$$

and we called them L, M and N, respectively, but now let's call them $S(1)$, $P(1)$, and $S(2)$ respectively. This naming will be explained in the next chapter. In order for this sequence to split as in Definition 1.5.2, we would need a map $h : {}^{1}_{2} \to 2$ such that $h \circ f = 1_2$. Let's compute $\mathrm{Hom}({}^{1}_{2}, 2)$:

$$\begin{tikzcd}
\Bbbk \arrow[r, "1"] \arrow[d, "x"'] & \Bbbk \arrow[d, "y"] \\
0 \arrow[r, "0"] & \Bbbk
\end{tikzcd}$$

Since x is a map to the zero vector space, $x = 0$. Let $z \in \Bbbk$ be in the top left vector space of the square. Following the top of the square, we have $z \mapsto z \mapsto yz$. Following the bottom we have $z \mapsto 0 \mapsto 0$. Therefore we need $yz = 0z$ and thus we need $y = 0$. We conclude that $\operatorname{Hom}(\begin{smallmatrix}1\\2\end{smallmatrix}, 2) = 0$, and moreover, that there can't exist such an h because there are no nontrivial morphisms from $\begin{smallmatrix}1\\2\end{smallmatrix}$ to 2. We have shown that f is not a section and therefore that $0 \to 2 \to \begin{smallmatrix}1\\2\end{smallmatrix} \to 1 \to 0$ does *not* split. Now let's apply the covariant Hom functor $\operatorname{Hom}(S(1), -)$ and get the exact sequence

$$0 \to \operatorname{Hom}(S(1), S(2)) \xrightarrow{f_*} \operatorname{Hom}(S(1), P(1)) \xrightarrow{g_*} \operatorname{Hom}(S(1), S(1)).$$

This sequence is exact because of Theorem 1.6.1; however, we claim that

$$0 \to \operatorname{Hom}(S(1), S(2)) \xrightarrow{f_*} \operatorname{Hom}(S(1), P(1)) \xrightarrow{g_*} \operatorname{Hom}(S(1), S(1)) \to 0$$

is *not* exact because g_* is not a surjection. To see this, we can (and should!) compute that $\operatorname{Hom}(S(1), P(1)) = 0$ and $\operatorname{Hom}(S(1), S(1)) \cong \Bbbk$. Thus g_* is a linear transformation from a finite dimensional vector space of dimension 0, to a finite dimensional vector space of dimension 1. Any linear transformation from a lower dimensional vector space to a higher dimensional one can be injective but *can not* be surjective (why?). This example illuminates the need for the splitting assumption in Corollary 1.6.4.

A natural question to ask here is, can we extend this sequence further to the right even when g_* isn't surjective? The answer to this question is yes! But to do this, we will need new ideas called extension functors, $\operatorname{Ext}^i(X, -)$ and $\operatorname{Ext}^i(-, X)$. We will see these at the end of the next chapter. We end this section with a little more abstract nonsense. Both Theorem 1.6.1 and Corollary 1.6.4 involve the covariant Hom functor. We have *dual* statements for the contravariant Hom functor (flip the arrows!)

Theorem 1.6.3. *Let $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a sequence of objects and morphisms in an abelian $\Bbbk$-category $\mathscr{C}$. Then it is exact if and only if*

$$Hom(L, X) \xleftarrow{f^*} Hom(M, X) \xleftarrow{g^*} Hom(N, X) \leftarrow 0$$

is exact for every $X \in Ob(\mathscr{C})$.

Exercise 1.6.2. Prove the previous theorem.

Corollary 1.6.4. *In an abelian $\Bbbk$-category $\mathscr{C}$, a sequence*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

splits if and only if for every $X \in Ob(\mathscr{C})$, the following sequence is exact:

$$0 \leftarrow Hom(L, X) \xleftarrow{f^*} Hom(M, X) \xleftarrow{g^*} Hom(N, X) \leftarrow 0.$$

Moreover, if

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

splits then so does

$$0 \leftarrow Hom(L, X) \xleftarrow{f^*} Hom(M, X) \xleftarrow{g^*} Hom(N, X) \leftarrow 0.$$

2 Auslander-Reiten Translation

Throughout this chapter, unless otherwise specified, let Q denote an acyclic n-vertex quiver. That is, Q has n vertices and no loops or cycles. A **cycle** is a path (Definition 1.1.3) in the quiver that starts and ends at the same vertex. A **loop** is a cycle of length 1.

2.1 Projectives, Injectives, and Simples in rep$Q$

Question 2.1. Why did we use those naming conventions in Example 1.6.3?

We will spend this section answering this question. Recall that our goal is to classify all isoclasses of indecomposable representations and the morphisms between them. To do this, it would be helpful to further classify the indecomposable representations into three classes: projective, injective, and simple representations. This classification is certainly not exhaustive, there are representations of some quivers that are neither projective, injective, nor simple. To explain what makes these representations ‘projective’, or ‘injective’, or ‘simple’, we will need more category theory and after the last section, it’s probably best to push that off for a little. We will describe what these words mean in the next section.

Definition 2.1.1. We define the **simple representation of Q at vertex** i, denoted by $S(i)$, as the representation whose morphisms and vector spaces are all zero, with the only exception that the vector space at the ith vertex is $\Bbbk$.

It follows from the definition that each quiver has precisely $n = |Q_0|$ of these simple representations. Notice that these representations are indecomposable (try to decompose it and see what happens!).

Example 2.1.1. Consider the quiver Q given by

$$\begin{tikzcd} & & 3 \arrow[dl, "\gamma"'] \\ 1 \arrow[r, "\alpha"] & 2 \arrow[dr, "\beta"] & \\ & & 4 \end{tikzcd}$$

Then the simple at each vertex is given by

$$
S(1): \begin{tikzcd} & & 0 \arrow[dl, "0"'] \\ \Bbbk \arrow[r, "0"] & 0 \arrow[dr, "0"] & \\ & & 0 \end{tikzcd}
\qquad
S(2): \begin{tikzcd} & & 0 \arrow[dl, "0"'] \\ 0 \arrow[r, "0"] & \Bbbk \arrow[dr, "0"] & \\ & & 0 \end{tikzcd}
$$

$$
S(3): \begin{tikzcd} & & \Bbbk \arrow[dl, "0"'] \\ 0 \arrow[r, "0"] & 0 \arrow[dr, "0"] & \\ & & 0 \end{tikzcd}
\qquad
S(4): \begin{tikzcd} & & 0 \arrow[dl, "0"'] \\ 0 \arrow[r, "0"] & 0 \arrow[dr, "0"] & \\ & & \Bbbk \end{tikzcd}
$$

We use the shorthand $S(1) = 1, S(2) = 2$, and so on.

The definitions of projectives and injectives are more complicated.

Definition 2.1.2. The **projective representation of Q at vertex** i is the representation $P(i) = (P(i)_j, \phi_\alpha)$ defined as follows.

The vector space $P(i)_j$ is the one whose basis consists of all paths *from i to j*. In other words, when we are constructing the projective at vertex i, we look at all the paths *out* of i.

If $\alpha : j \to l$ is an arrow in Q from j to l, then the morphism $\varphi_\alpha : P(i)_j \to P(i)_l$ is the morphism defined on the basis by composing the paths from i to j with α. More precisely, let $p \in P(i)_j$ be a path from i to j. So $p = \beta_1\beta_2 \dots \beta_n$ where $s(\beta_1) = i$ and $t(\beta_n) = j$. Then $\varphi_\alpha(p) = p\alpha = \beta_1\beta_2 \dots \beta_n\alpha$. This new path starts at $s(\beta_1) = i$ and ends at $t(\alpha) = l$. Thus $p\alpha \in P(i)_l$.

The morphism φ_α gives an injection from the basis of $P(i)_j$ to that of $P(i)_l$. We extend φ_α linearly to the entire vector space $P(i)_j$ as in Proposition 1.1.1; that is, an arbitrary element of $P(i)_j$ is a linear combination of the basis elements, which are the paths from i to j. Thus an arbitrary element of $P(i)_j$ can be written as $\sum_p \lambda_p p$ where we are summing over all paths p in the basis of $P(i)_j$ and the $\lambda_p \in \Bbbk$ are scalars. We define

$$\varphi_\alpha(\sum_p \lambda_p p) = \sum_p \lambda_p \varphi_\alpha(p) = \sum_p \lambda_p p\alpha.$$

Example 2.1.2. Let Q be the quiver from Example 2.1.1. Let's find $P(1)$. We begin by computing the vector spaces. The first one is $P(1)_1$, which is the vector space of all paths from 1 to 1. There is precisely one of these, namely the lazy path e_1. Thus a basis for $P(1)_1 = \{e_1\}$ and we conclude that $P(1)_1$ is a one dimensional $\Bbbk$ vector space, hence $P(1)_1 \cong \Bbbk$. Now let's compute $P(1)_2$. There is only one path from 1 to 2, namely α. There is technically two other paths $e_1\alpha$ and αe_2, however, recall that both of these paths are the same as α in the algebra $\Bbbk Q$. Therefore a basis for $P(1)_2$ is $\{\alpha\}$, so $P(1)_2 \cong \Bbbk$. Similarly, a basis for $P(1)_4$ is $\{\alpha\beta\}$, so

$P(1)_4 \cong \Bbbk$. The one vertex that is different is vertex 3. Notice that if we start at one and follow α, there is no way to get to vertex 3 since γ starts at vertex 3 and ends at vertex 2. Therefore, there is no path from vertex 1 to vertex 3. We conclude that $P(1)_3 = 0$. Thus we have the vector spaces for $P(1)$:

$$\begin{tikzcd}
& & 0 \arrow[dl, "\varphi_\gamma"'] \\
\Bbbk \arrow[r, "\varphi_\alpha"] & \Bbbk \arrow[dr, "\varphi_\beta"] & \\
& & \Bbbk
\end{tikzcd}$$

We now need to compute the morphisms. We will define these on the bases. We have that φ_α sends $e_1 \mapsto e_1\alpha = \alpha$. Thus φ_α sends the basis of $P(1)_1$ bijectively to the basis of $P(1)_2$, which is equivalent to the identity map between the two vector spaces $\Bbbk$. Similarly, φ_β sends $\alpha \mapsto \alpha\beta$. So φ_β maps the basis of $P(1)_2$ bijectively to the basis of $P(1)_4$ and hence, is equivalent to the identity map between the two vector spaces $\Bbbk$. Finally, φ_γ is the zero map since it is a linear transformation from the zero vector space. Therefore $P(1)$ is isomorphic to

$$\begin{tikzcd}
& & & 0 \arrow[dl, "0"'] \\
P(1) \cong & \Bbbk \arrow[r, "1"] & \Bbbk \arrow[dr, "1"] & \\
& & & \Bbbk
\end{tikzcd}$$

The reader should verify that the rest of the projectives at vertex i are given as follows.

$$\begin{tikzcd}
& & & 0 \arrow[dl, "0"'] \\
P(2) \cong & 0 \arrow[r, "0"] & \Bbbk \arrow[dr, "1"] & \\
& & & \Bbbk
\end{tikzcd}
\qquad
\begin{tikzcd}
& & & \Bbbk \arrow[dl, "1"'] \\
P(3) \cong & 0 \arrow[r, "0"] & \Bbbk \arrow[dr, "1"] & \\
& & & \Bbbk
\end{tikzcd}$$

$$\begin{tikzcd}
& & & 0 \arrow[dl, "0"'] \\
P(4) \cong & 0 \arrow[r, "0"] & 0 \arrow[dr, "0"] & \\
& & & \Bbbk
\end{tikzcd}$$

Remark 2.1.1. Notice that in 2.1.2, the only projective that is also simple is $P(4)$. This is because vertex 4 is a **sink** in the quiver. A vertex i in Q is called a **sink** if no arrows start at i. This holds in general! The projective at vertex i is simple, $P(i) = S(i)$, if and only if i is a sink in Q.

Let's do a more complicated example.

Example 2.1.3. Let Q be the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\gamma"] & \\ 1 \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & 4 \\ & 3 \arrow[ur, "\delta"'] & \end{tikzcd}$$

Let's compute $P(1)$. We have the following bases for the vector spaces at each vertex (check it!):

$$P(1)_1 : \{e_1\} \cong \Bbbk, \;\; P(1)_2 : \{\alpha\} \cong \Bbbk, \;\; P(1)_3 : \{\beta\} \cong \Bbbk, \;\; P(1)_4 : \{\alpha\gamma, \beta\delta\} \cong \Bbbk^2.$$

Therefore $P(1)$ is isomorphic to the following representation:

$$\begin{tikzcd} & \Bbbk \arrow[dr, "\varphi_\gamma"] & \\ \Bbbk \arrow[ur, "\varphi_\alpha"] \arrow[dr, "\varphi_\beta"'] & & \Bbbk^2 \\ & \Bbbk \arrow[ur, "\varphi_\delta"'] & \end{tikzcd}$$

It remains to find the morphisms. We have the following:

$$\{e_1\} \overset{\varphi_\alpha}{\mapsto} \{\alpha\} \implies \varphi_\alpha = 1 \text{ and } \{e_1\} \overset{\varphi_\beta}{\mapsto} \{\beta\} \implies \varphi_\beta = 1.$$

Now, $\{\alpha\} \overset{\varphi_\gamma}{\mapsto} \{\alpha\gamma\}$, so φ_γ is *not* a bijection, just an injection. Recall that the basis for $P(1)_4$ is $\{\alpha\gamma, \beta\delta\}$, so φ_γ sends the basis of $P(1)_2$ to the *first* basis element of $P(1)_4$. Thus $\varphi_\gamma = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Similarly,

$$\{\beta\} \overset{\varphi_\delta}{\mapsto} \{\beta\delta\} \implies \varphi_\delta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore $P(1)$ is isomorphic to

$$\begin{tikzcd} & \Bbbk \arrow[dr, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] & \\ \Bbbk \arrow[ur, "1"] \arrow[dr, "1"'] & & \Bbbk^2 \\ & \Bbbk \arrow[ur, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"'] & \end{tikzcd}$$

Wait! Does it matter the order in which we write the elements of the basis $\{\alpha\gamma, \beta\delta\}$?

Exercise 2.1.1. Show that if we made $\varphi_\gamma = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\varphi_\delta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the two representations would be isomorphic.

Exercise 2.1.2. Compute $P(2), P(3)$, and $P(4)$.

Dually, we can instead look at all the arrows *into* vertex i, and this would give us the next definition. Even though we aren't flipping any arrows in an obvious way (we will see how the arrows are flipped in the next section), we are flipping the way we think of the bases.

Definition 2.1.3. The **injective representation of Q at vertex** i is the representation $I(i) = (I(i)_j, \phi_\alpha)$ defined as follows.

The vector space $I(i)_j$ is the one whose basis consists of all paths *to* i *from* j. In other words, when we are constructing the injective at vertex i, we look at all the paths *into* i.

If $\alpha : j \to l$ is an arrow in Q from j to l, then the morphism $\varphi_\alpha : I(i)_j \to I(i)_l$ is the morphism defined on the basis by 'deleting' the arrow α from all paths from j to i which start at α. If the path does not start at α, φ_α sends it to zero. More precisely, let $p \in I(i)_j$ be a path from j to i. So $p = \beta_1\beta_2 \dots \beta_n$ where $s(\beta_1) = j$ and $t(\beta_n) = i$. Then

$$\varphi_\alpha(p) = \begin{cases} e_l\beta_2 \dots \beta_n = \beta_2 \dots \beta_n & \beta_1 = \alpha \\ 0 & \text{otherwise} \end{cases}$$

This new path starts at $s(\beta_2) = l$ and ends at $t(\alpha) = i$. Thus $\beta_2 \dots \beta_n \in I(i)_l$.

Wait! Why is 0 in $I(i)_l$ if it isn't a path from l to i?

The morphism φ_α gives a surjection from the basis of $I(i)_j$ to that of $I(i)_l$. We extend φ_α linearly to the entire vector space $I(i)_j$ as in Definition 2.1.2; that is,

$$\varphi_\alpha(\sum_p \lambda_p p) = \sum_p \lambda_p \varphi_\alpha(p)$$

Example 2.1.4. Let's look at the quiver from Examples 2.1.1 and 2.1.2. Let's compute $I(2)$ by first computing the vector spaces. The first one is $I(2)_1$, which is the vector space of all paths from 1 to 2. There is precisely one of these, namely α. Thus a basis for $I(2)_1 = \{\alpha\}$ and we conclude that $I(2)_1$ is a one dimensional $\Bbbk$ vector space, hence $I(2)_1 \cong \Bbbk$. Now let's compute $I(2)_2$. There is only one path from 2 to 2, namely e_2. Therefore a basis for $I(2)_2$ is $\{e_2\}$, so $I(2)_2 \cong \Bbbk$. Similarly, a basis for $I(2)_3$ is $\{\gamma\}$, so $I(2)_3 \cong \Bbbk$. The one vertex that is different is vertex 4. Notice that if we start at 4, there is no way to get to vertex 2 since γ starts at vertex 2 and ends at vertex 4. Therefore, there is no path from vertex 4 to vertex 2. We conclude that $I(2)_4 = 0$. Thus we have the vector spaces for $I(1)$:

$$\begin{tikzcd} & & \Bbbk \arrow[dl, "\varphi_\gamma"'] \\ \Bbbk \arrow[r, "\varphi_\alpha"] & \Bbbk \arrow[dr, "\varphi_\beta"] & \\ & & 0 \end{tikzcd}$$

We now need to compute the morphisms. We will define these on the bases. Since α is the only element in the basis of $I(2)_1$, we only need to see what φ_α does to α. Note that α is a path from 1 to 2 that starts with α, so $\varphi_\alpha(\alpha) = e_2$. Thus φ_α sends the basis of $I(2)_1$ bijectively to the basis of $I(2)_2$, which is equivalent to the identity map between the two vector spaces $\Bbbk$. Similarly, φ_γ sends $\gamma \mapsto e_2$. So φ_γ maps the basis of $I(2)_3$ bijectively to the basis of $I(2)_2$ and hence, is equivalent to the identity map between the two vector spaces $\Bbbk$. Finally, φ_β is the zero map since it is a linear transformation to the zero vector space. Therefore $I(2)$ is isomorphic to

$$I(2) \cong \begin{tikzcd} & & \Bbbk \arrow[dl, "1"'] \\ \Bbbk \arrow[r, "1"] & \Bbbk \arrow[dr, "0"] & \\ & & 0 \end{tikzcd}$$

The reader should verify that the rest of the injectives at vertex i are given as follows.

$$I(1) \cong \begin{tikzcd} & & 0 \arrow[dl, "0"'] \\ \Bbbk \arrow[r, "0"] & 0 \arrow[dr, "0"] & \\ & & 0 \end{tikzcd} \qquad I(3) \cong \begin{tikzcd} & & \Bbbk \arrow[dl, "0"'] \\ 0 \arrow[r, "0"] & 0 \arrow[dr, "0"] & \\ & & 0 \end{tikzcd}$$

$$I(4) \cong \begin{tikzcd} & & \Bbbk \arrow[dl, "1"'] \\ \Bbbk \arrow[r, "1"] & \Bbbk \arrow[dr, "1"] & \\ & & \Bbbk \end{tikzcd}$$

Remark 2.1.2. Notice that in 2.1.4, the only injectives that are also simple are $I(1)$ and $I(3)$. This is because vertices 1 and 3 are each a **source** in the quiver. A vertex i in Q is called a **source** if no arrows end at i. This holds in general! The injective at vertex i is simple, $I(i) = S(i)$, if and only if i is a source in Q.

Time for a slightly more complicated example:

Example 2.1.5. Let Q be the quiver from Example 2.1.3. Let's compute $I(4)$. We have the following bases for the vector spaces at each vertex (check it!):

$$I(4)_1 : \{\alpha\gamma, \beta\delta\} \cong \Bbbk^2, \;\; I(4)_2 : \{\gamma\} \cong \Bbbk, \;\; I(4)_3 : \{\delta\} \cong \Bbbk, \;\; I(4)_4 : \{e_4\} \cong \Bbbk.$$

Therefore $I(4)$ is isomorphic to the following representation:

$$\begin{tikzcd}
 & \Bbbk \arrow[dr, "\varphi_\gamma"] & \\
\Bbbk^2 \arrow[ur, "\varphi_\alpha"] \arrow[dr, "\varphi_\beta"'] & & \Bbbk \\
 & \Bbbk \arrow[ur, "\varphi_\delta"'] &
\end{tikzcd}$$

It remains to find the morphisms. We have the following:

$$\{\gamma\} \overset{\varphi_\gamma}{\mapsto} \{e_4\} \implies \varphi_\gamma = 1 \text{ and } \{\delta\} \overset{\varphi_\delta}{\mapsto} \{e_4\} \implies \varphi_\delta = 1.$$

Now, $\{\alpha\gamma, \beta\delta\} \overset{\varphi_\alpha}{\mapsto} \{\gamma, 0\}$, so φ_α is *not* a bijection, just a surjection. Recall that the basis for $I(4)_1$ is $\{\alpha\gamma, \beta\delta\}$, so φ_α sends the *first* basis element of $I(4)_1$ to the basis element of $I(4)_2$ and the other element to 0. Thus $\varphi_\alpha = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Similarly,

$$\{\alpha\gamma, \beta\delta\} \overset{\varphi_\beta}{\mapsto} \{0, \delta\} \implies \varphi_\delta = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Therefore $I(4)$ is isomorphic to

$$\begin{tikzcd}
 & \Bbbk \arrow[dr, "1"] & \\
\Bbbk^2 \arrow[ur, "{\begin{bmatrix} 1 & 0 \end{bmatrix}}"] \arrow[dr, "{\begin{bmatrix} 0 & 1 \end{bmatrix}}"'] & & \Bbbk \\
 & \Bbbk \arrow[ur, "1"'] &
\end{tikzcd}$$

Exercise 2.1.3. Show that if we made $\varphi_\alpha = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $\varphi_\beta = \begin{bmatrix} 1 & 0 \end{bmatrix}$, the two representations would be isomorphic.

Exercise 2.1.4. Compute $I(1), I(2)$, and $I(3)$.

Remark 2.1.3. Notice that we need the assumption that Q has no cycles. For instance consider the quiver $Q : 1 \rightleftarrows 2$ with top arrow α and bottom arrow β. If we want to compute the projective at one, then $P(1)_1$ is the vector space with basis the set of all paths from 1 to 1: $\{e_1, \alpha\beta, \alpha\beta\alpha\beta, \dots\}$. This is an infinite dimensional vector space and we have immediately left rep$Q$, the category of finite dimensional representations of Q.

The reader at this point may have noticed that all these representations are indecomposable (if not, go notice it!). This is indeed the case and we will finish this section proving this.

Proposition 2.1.1. *The representation $S(i), P(i)$, and $I(i)$ are indecomposable.*

Proof. For the simples, this follows from the fact that $S(i)$ has no subrepresentations other than itself and 0. Therefore, there can't be any 'smaller' representations that add up to $S(i)$ other than $S(i) \oplus 0$, which is not a decomposition as in Definition 1.3.2. Now let's prove this for the projective $P(i) = (P(i)_j, \varphi_\alpha)$. We will use a contradiction. Suppose $P(i) = M \oplus N$ for some finite dimensional representations M and N such that neither are 0 nor $P(i)$. Since we assume Q to have no oriented cycles, the only way to get from a vertex to itself is by taking the lazy path. That means that $P(i)_i \cong \Bbbk$ since it has a one dimensional basis $\{e_i\}$. This means that if we add M and N together, the direct sum of their ith vector spaces must be $\Bbbk$ which is one dimensional. This implies that one of them must be the zero space and the other must be one dimensional by the pigeon hole principle. We need to pick one, so let's say $M_i \cong \Bbbk$ and $N_i = 0$. But we assume that N is not the zero representation, so that means there must be some vertex in the representation at which the vector space is not zero. Suppose this happens at vertex l, so $N_l \neq 0$. We will contradict this fact.

The vector space $P(i)_l$ has a basis consisting of all paths from i to l. Let's pick a path $p = \beta_1\beta_2 \dots \beta_s$ from i to l. Then corresponding to this path in the representation $P(i)$, we have:

$$\Bbbk^{i_1} \xrightarrow{\varphi_{\beta_1}} \Bbbk^{i_2} \xrightarrow{\varphi_{\beta_2}} \Bbbk^{i_3} \xrightarrow{\varphi_{\beta_3}} \dots \xrightarrow{\varphi_{\beta_s}} \Bbbk^{i_{s+1}}.$$

We can in general define the morphism for the path p by following (composing) all the morphisms corresponding to each arrow in the path; that is, $\varphi_p = \varphi_{\beta_s} \circ \dots \circ \varphi_{\beta_1}$. If we feed φ_p the unique basis element at vertex i, namely e_i, then we get:

$$\varphi_p(e_i) = \varphi_{\beta_s}(\dots(\varphi_{\beta_1}(e_i)) = \varphi_{\beta_s}(\dots(\varphi_{\beta_2}(e_i\beta_1)) = \varphi_{\beta_s}(\dots(\varphi_{\beta_3}(e_i\beta_1\beta_2)) = \dots = \beta_1\beta_2\dots\beta_s = p.$$

On the other hand, this map φ_p defines a map from $P(i)_i$ to $P(i)_l$, so by assumption we have a map

$$\varphi_p : M_i \oplus 0 \xrightarrow{\begin{bmatrix} \varphi_p & 0 \\ 0 & 0 \end{bmatrix}} M_l \oplus N_l$$

that sends the unique basis element e_i of M_i to an element $\varphi_p(e_i) = p \in M_l$. But since we can do this *for any* path p, this means that all paths from i to l are contained in M_l. In other words, the entire basis of $P(i)_l$ is contained in M_l and since $P(i)_l = M_l \oplus N_l$ by assumption, this forces $N_l = 0$. But this is a contradiction! We assumed that $N_l \neq 0$, so therefore our original assumption is false. Namely, $P(i) \neq M \oplus N$ for some finite dimensional representations M and N such that neither are 0 nor $P(i)$, and we conclude that $P(i)$ is indecomposable for each i. □

Exercise 2.1.5. Show that $I(i)$ is indecomposable.

2.2 Projectives, Injectives, and Simples in a Category

Question 2.2. What does it mean to be projective, injective, or simple?

The notion of projectivity, injectivity, and simplicity is very important and appears frequently in mathematics. Of course, when an idea appears in a bunch of seemingly disparate branches of mathematics, there must be some categorical notion that connects them all. The following definition holds in any category, but then we would need the notions of epimorphisms and monomorphisms. In an abelian category, these are the same as surjective morphisms and injective morphisms respectively.

Zooming Out 9. *Let $\mathscr{C}$ be an abelian $\mathbb{k}$-category.*

1. *An object $S \in Ob(\mathscr{C})$ is* ***simple*** *if and only if its only subobjects are S and 0. We have already seen that the simple at vertex i is a simple object in rep$Q$.*

2. *An object $P \in Ob(\mathscr{C})$ is* ***projective*** *if and only if for any surjective morphism $g : M \to N$ and morphism $f : P \to N$, there is a morphism $h : P \to M$ such that $g \circ h = f$. Here is a picture:*

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dotted, "h"'] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

   This diagram commuting is equivalent to saying for every $f \in Hom(P, N)$, there exists some $h \in Hom(P, M)$ such that $g_*(h) = g \circ h = f$. This is precisely the definition of surjectivity of g_*! Thus an object P in $\mathscr{C}$ is projective if and only if for any surjective morphism $g : M \to N$, the map

$$g_* : Hom(P, M) \to Hom(P, N)$$

   is surjective. That is, the functor $Hom(P, -)$ takes surjective morphisms to surjective morphisms $g \mapsto Hom(P, g) = g_*$.

3. *An object $I \in Ob(\mathscr{C})$ is* ***injective*** *if and only if for any injective morphism $g : N \to M$ and morphism $f : N \to I$, there is a morphism $h : M \to I$ such that $h \circ g = f$. Here is a picture:*

$$\begin{tikzcd} & I & \\ M \arrow[ur, dotted, "h"] & N \arrow[l, "g"] \arrow[u, "f"'] & 0 \arrow[l] \end{tikzcd}$$

   Aha! We said we would flip the arrows! This diagram commuting is equivalent to saying for every $f \in Hom(N, I)$, there exists some $h \in Hom(M, I)$ such that $g^*(h) = h \circ g = f$. This is precisely the definition of surjectivity of g^*! Thus an object I in $\mathscr{C}$ is injective if and only if for any injective morphism $g : N \to M$, the map

$$g^* : Hom(M, I) \to Hom(N, I)$$

   is surjective. That is, the functor $Hom(-, I)$ takes injective morphisms to surjective morphisms $g \mapsto Hom(g, I) = g^*$.

Okay, now it's time to see why the projective at vertex i is actually a projective representation in rep$Q$.

Proposition 2.2.1. *The projective at vertex i, $P(i)$, is a projective representation in rep$Q$.*

Proof. In order to prove this, we need to show that the first diagram in Zooming Out 9 commutes. So let $g : M \to N$ be a surjective morphism in rep$Q$ and let $f : P(i) \to N$ be any map. Then we need to construct a map $h : P(i) \to M$ such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, dotted, "h"'] \arrow[d, "f"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

To do this, we will rely on the structure of the projective at the vertex i. Let $P(i) = (P(i)_j, \varphi_\alpha), M = (M_j, \psi_\alpha)$, and $N = (N_j, \xi_\alpha)$. Since the vector space $P(i)_j$ has a basis consisting of all arrow from i to j, we should begin defining h at the ith vector space. Because we want $g_i \circ h_i = f_i$, we need to analyze the map f_i. We know that $P(i)_i$ is the vector space with basis $\{e_i\}$ since we assume Q to have no loops or cycles. Thus f_i is defined entirely by where it sends e_i. Suppose $f_i(e_i) = n_i \in N_i$. Then since g is surjective, there exists at least one element $m_i \in M_i$ such that $g(m_i) = n_i$. There could be multiple of them from which we are picking one. The morphism h depends on this choice of m_i which means that this morphism h won't be unique. But that's okay! No where in the definition of projective object do we require this map to be unique, we just need to show there is at least one. Okay, so then lets define $h_i(e_i) = m_i$. Then we have $g_i(h_i(e_i)) = g_i(m_i) = n_i = f(e_i)$, so $g_i \circ h_i = f_i$ as needed.

Now we need to define h_j for all the other vertices j and this is where the structure of $P(i)$ comes into play. We know that the basis of $P(i)_j$ is $\{p_k : p_k \text{ is a path from } i \text{ to } j \text{ in } Q\}$. Whence let $p_k \in P(i)_j$, so $p_k = \beta_1\beta_2 \dots \beta_l$ where $s(\beta_1) = i$ and $t(\beta_l) = j$. Then in the representation M, we have a composition of the morphisms corresponding to this path, namely $\psi_{\beta_l} \circ \cdots \circ \psi_{\beta_1}$. We define h_j by *following the paths in Q*; that is, $h_j(\beta_1\beta_2 \dots \beta_l) = \psi_{\beta_l}(\dots(\psi_{\beta_1}(m_i))\dots)$.

Wait! Why did we define h_j in this seemingly strange way?

Now that we have defined the map h, we leave the fact that h is well defined to the reader.

Exercise 2.2.1. Show that $h : P(i) \to M$ is a well defined morphism of representations (ie show the boxes commute).

Now we need to show that $g \circ h = f$. To do this, consider the diagram below. Using the commutivity of the boxes, along with the definition of h, we leave this as an exercise to the reader.

Exercise 2.2.2. Show that $g \circ h = f$.

$$\begin{tikzcd}
P(i)_j \arrow[r, "\varphi_\alpha"] \arrow[d, "h_j"'] & P(i)_k \arrow[d, "h_k"] \\
M_j \arrow[r, "\psi_\alpha"] \arrow[d, "g_j"'] & M_k \arrow[d, "g_k"] \\
N_j \arrow[r, "\xi_\alpha"] & N_k \\
P(i)_j \arrow[r, "\varphi_\alpha"] \arrow[u, "f_j"] & P(i)_k \arrow[u, "f_k"']
\end{tikzcd}$$

□

Dually, we have the following proposition.

Proposition 2.2.2. *The injective at vertex i is an injective object in rep$Q$.*

Exercise 2.2.3. Prove the previous proposition.

One reason we like projective and injective objects is the following proposition, which holds in particular for representations of quivers.

Proposition 2.2.3.

1. *Let P be a projective object in an abelian $\Bbbk$-category $\mathscr{C}$. Then any exact sequence of the form*
$$0 \to L \to M \xrightarrow{g} P \to 0$$
splits.

2. *Let I be an injective object in an abelian $\Bbbk$-category $\mathscr{C}$. Then any exact sequence of the form*
$$0 \to I \xrightarrow{g} M \to N \to 0$$
splits.

Proof. We will prove 1. and to do this, we will show that g is a retraction which would show that the sequence splits by Definition 1.5.2. Consider the identity morphism $1_P : P \to P$. Then since P is projective, by definition, we have the commutative diagram:

$$\begin{tikzcd}
 & P \arrow[dl, dotted, "h"'] \arrow[d, "1_P"] & \\
M \arrow[r, "g"'] & P \arrow[r] & 0
\end{tikzcd}$$

Thus $1_P = g \circ h$ and g is a retraction. □

Exercise 2.2.4. Prove the proposition for injectives.

Another useful fact about projectivity and injectivity is that it is preserved under direct sums!

Proposition 2.2.4.

1. *If P and P' are objects in an abelian $\Bbbk$-category $\mathscr{C}$, then $P \oplus P'$ is a projective object if and only if both P and P' are projective.*

2. *If I and I' are objects in an abelian $\Bbbk$-category $\mathscr{C}$, then $I \oplus I'$ is an injective object if and only if both I and I' are injective.*

Proof. We will prove 1. Suppose $P \oplus P'$ is a projective object. To show P is projective, let $f : P \to N$ be a morphism and $g : M \to N$ a surjective morphism. Notice, we have a split exact sequence $0 \to P' \xrightarrow{i} P \oplus P' \xrightarrow{p} P \to 0$ where i is the inclusion and p is the projection. Since this sequence is split, p is a retraction and we have a morphism $h' : P \to P \oplus P'$ such that $p \circ h' = 1_P$. Also, we have a map from $P \oplus P'$ to N given by the composition $P \oplus P' \xrightarrow{p} P \xrightarrow{f} N$. Thus we have the following diagram:

$$\begin{tikzcd}
& P \oplus P' \arrow[d, "p"] \arrow[ddl, dotted, bend right, "h"'] & \\
& P \arrow[u, bend right, "h'"'] \arrow[dl, "h \circ h'"'] \arrow[d, "f"] & \\
M \arrow[r, "g"] & N \arrow[r] & 0
\end{tikzcd}$$

The existence of the morphism h such that $g \circ h = f \circ p$ follows from the projectivity of $P \oplus P'$. Then we have a morphism $h \circ h' : P \to M$ such that $(g \circ h) \circ h' = f \circ (p \circ h') = f \circ 1_p = f$. Therefore P is projective and the proof for P' is analogous.

Conversely, suppose that both P and P' are projective objects, that $g : M \to N$ is a surjective morphism, and $f : P \oplus P' \to N$ is any morphism. Then we have the two following diagrams.

$$\begin{tikzcd}
& P \arrow[d, "i_1"] \arrow[ddl, dotted, bend right, "h_1"'] & \\
& P \oplus P' \arrow[u, bend right, "p_1"'] \arrow[d, "f"] & \\
M \arrow[r, "g"] & N \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
& P' \arrow[d, "i_2"] \arrow[ddl, dotted, bend right, "h_2"'] & \\
& P \oplus P' \arrow[u, bend right, "p_2"'] \arrow[d, "f"] & \\
M \arrow[r, "g"] & N \arrow[r] & 0
\end{tikzcd}$$

The morphisms labeled h_1 and h_2 exist because P and P' are projective. The maps i_1, i_2, p_1, and p_2 exist from similar short exact sequences as the one in the first paragraph.

Wait! Write down the sequences from which the maps i_1, i_2, p_1, and p_2 arise!

We need to construct a morphism h from $P \oplus P'$ to M that makes the triangle commute. In general, this follows from the categorical definition of direct sum, which we have not and will not define, but here we will assume that P, P', M, and N are all representations of some quiver Q. We will define this map h by combining both the maps h_1 and h_2. Define

$h(p + p') = h_1(p) + h_2(p')$. (The sum of two things may not even make any sense in M when $\mathscr{C}$ is an arbitrary abelian $\Bbbk$-category!)

Exercise 2.2.5. Show that this map is a well defined morphism in rep$Q$ and that $g \circ h = f$.

□

This theorem is really quite useful for us! If we know all the indecomposable projectives and injectives, then we know all the projectives and injectives because we can just add them together. We do know that the projective/injective at vertex i is an indecomposable projective/injective representation in rep$Q$, but we don't know that this is *all* of the indecomposable projectives/injective representations. If we want to classify all indecomposable representations of Q, then we certainly will need to find all indecomposable projectives and injectives. Actually, the projective/injectives at vertex i are all of the indecomposable projective/injectives which we will prove in the next section. But for now, we can show that the collection of $S(i)$ forms a complete collection of simples in rep$Q$. To do this, we begin with a lemma.

Lemma 2.2.5. *For any indecomposable representation M in rep$Q$, there exists a vertex i such that $S(i)$ is a subrepresentation of M.*

Proof. We will do this by very strategically choosing the vertex i. Since we assume throughout this section that Q does not have any cycles, there must be in some sense 'a last nonzero vector space' in the representation. What we mean is that there must exist a vertex i at which there are either no linear transformations out of M_i (i is a sink in Q), or all of the linear transformations out of M_i are zero. Suppose there does not exists such a vertex. Then at each vertex, there is a nontrivial linear transformation *out of* that vertex. But since there are only finitely many vertices, this means there is no sink in the quiver Q, which means Q has a cycle, a contradiction.

So suppose i is a sink or the vertex at which all linear transformations out of M_i are 0. Then at vertex i, we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"] & \Bbbk \arrow[r, "0"] \arrow[d, "f_i"] & 0 \arrow[d, "0"] \\
M_l \arrow[r, "\varphi_\alpha"] & M_i \arrow[r, "0"] & 0
\end{tikzcd}$$

Since M_i is a nontrivial vector space, its dimension is at least one, in which case there is an injection from the one dimensional vector space $\Bbbk$ into M_i. We define the linear transformation $f_i : \Bbbk \to M_i$ to be the inclusion. Define a morphism $f : S(i) \to M_i$ as $(0, \ldots, 0, f_i, 0, \ldots, 0)$. Then f gives an injective morphism from $S(i)$ to M, whence $S(i)$ is a subrepresentation of M as desired. □

Theorem 2.2.6. *A representation of Q is simple if and only if it is isomorphic to $S(i)$ for some vertex i in Q_0.*

Proof. We have already seen the reverse direction. If an object is isomorphic to $S(i)$, then it must be simple since $S(i)$ is simple (only has $S(i)$ and 0 as subrepresentations).

On the other hand, suppose we have a simple object M in rep$Q$. Then by Lemma 2.2.5, there exists a vertex i such that $S(i)$ is a subrepresentation of M. Since M is simple, M can't have any subrepresentation other than M and 0, which forces $M \cong S(i)$. □

Exercise 2.2.6. Write down some other simple representations for the one loop quiver.

$$\begin{tikzcd} 1 \arrow[loop right, "\alpha"] \end{tikzcd}$$

We finish with two more properties of projective objects that will be very useful when we begin constructing the so-called 'Nakayama functor'. The first theorem says that the vector space at vertex i of any representation of Q can be realized as the vector space of morphisms *out of* the projective at vertex i.

Theorem 2.2.7. *Let M be an object of rep$Q$. Then for any $i \in Q_0$,*

$$Hom(P(i), M) \cong M_i.$$

Proof. Before we begin, let's give names to our representations. Let $M = (M_i, \psi_\alpha)$ and $P(i) = (P(i)_j, \varphi_\alpha)$. To show the above isomorphism, we will construct a linear transformation ϕ between the vector spaces $\mathrm{Hom}(P(i), M)$ and M_i. Then we will show that this linear transformation is a bijection, hence an isomorphism. So let $\phi : \mathrm{Hom}(P(i), M) \to M_i$ be defined by $\phi(f = (f_j)) = f_i(e_i)$. That is, we are *defining* ϕ just by where it sends the basis $\{e_i\}$ of $P(i)_i$ and ignoring all the other linear transformations f_j.

Exercise 2.2.7. Show that ϕ is a well-defined linear transformation.

Now we will show that ϕ is injective. To do this, we will show that it has trivial kernel; that is, the only morphism from $P(i)$ to M that ϕ sends to zero is the zero morphism. Whence suppose $\phi(f) = f_i(e_i) = 0$. Then at vertex i, we have that f_i sends the entire basis of $P(i)_i$ to zero, making f_i the zero map. But this does not mean that f is the zero morphism in general! We also need to show that $f_j = 0$ for all the other maps! To do this, we will rely on the structure of $P(i)$. Let j be a different vertex in Q and consider the linear transformation $f_j : P(i)_j \to M_j$. We know that $P(i)_j$ has basis $\{p_1, p_2, \ldots, p_k\}$ where p_l is a path from i to j in Q for each l. Choose one of these basis elements and call it $p = \beta_1\beta_2 \ldots \beta_m$ where $s(\beta_1) = i$ and $t(\beta_m) = j$. Then by *following the paths in* Q, we have morphisms $\varphi_p = \varphi_{\beta_m} \circ \cdots \circ \varphi_{\beta_1} : P(i)_i \to P(i)_j$ and $\psi_p : M_i \to M_j$. Using the definition of $P(i)$, as we've seen in the proof of Proposition 2.1.1, we have that $\varphi_p(e_i) = p$. Since f is a morphism of representations by assumption, every box for each β in the path p commutes, so we have that $f_j \circ \varphi_p = \psi_p \circ f_i$. Since $f_i(e_i) = 0$ as shown in the beginning of this paragraph, we have $f_j(p) = f_j(\varphi_p(e_i)) = \psi_p(f_i(e_i)) = \psi_p(0) = 0$. Thus $f_j(p) = 0$ and since this holds for all p, f_j sends the basis of $P(i)_j$ to 0, making $f_j = 0$. Therefore f is the zero morphism.

Finally, to show surjectivity, we need to choose an element $m_i \in M_i$, then construct a morphism $f \in \mathrm{Hom}(P(i), M)$ such that $\phi(f) = m_i$. We do this by first specifying that $f_i(e_i) = m_i$, so we've defined the linear transformation at vertex i.

Exercise 2.2.8. Finish constructing the morphism f by following the paths in Q.

Therefore, ϕ is a bijective linear transformation between the two vector spaces, hence $\mathrm{Hom}(P(i), M) \cong M_i$. □

An immediate consequence of this theorem is the following.

Corollary 2.2.8.

1. *The vector space Hom$(P(i), P(j))$ has a basis consisting of all paths from j to i in Q, which implies*
$$\mathit{End}(P(i)) := \mathit{Hom}(P(i), P(i)) \cong \Bbbk.$$

2. *If $A = \oplus_{i \in Q_0} P(i)$, then the vector space End$(A) =$ Hom(A, A) has a basis consisting of all paths in Q.*

Exercise 2.2.9. Use the previous theorem to prove the Corollary.

Wait a second, part 2. of the above Corollary looks familiar. Recall the path algebra $\Bbbk Q$ from Definition 1.1.4. This is the algebra that has basis all paths in Q! Well, we can multiply two morphisms together, $fg(x) = f(x)g(x)$, and this multiplication satisfies all the axioms in Definition 1.1.1, so $\mathrm{End}(A)$ is actually an algebra! If the two algebras have the same basis, then as long as multiplication coincides, they are isomorphic! This means that we can *completely* describe the path algebra of a quiver by just looking at the direct sum of all the projectives at vertex i! Pretty neat. Without proving the result, we finish the section by stating it here.

Theorem 2.2.9. *Let Q be a quiver. Then we have the following isomorphism of algebras*

$$\Bbbk Q \cong \mathit{End}(A)$$

where $A = \oplus_{i \in Q_0} P(i)$ is the direct sum of the projectives at vertex i.

2.3 Projective/Injective Resolutions

The last section was pretty abstract, so in this section let's do more examples. We have already seen how we can use the projectives to understand the vector space at vertex i of any representation M, Theorem 2.2.7. We have also seen how we can use projectives to understand the path algebra $\Bbbk Q$, Theorem 2.2.9. In this section, we will answer the following two questions.

Question 2.3. How can we use projectives to understand an arbitrary representation M? How do we know that $P(i)$ and $I(i)$ give a complete set of indecomposable projectives and injectives?

We begin with the first question.

Zooming Out 10. *Let $\mathscr{C}$ be an abelian $\mathbb{k}$-category and $M \in Ob(\mathscr{C})$.*

1. *A **projective resolution** of M is an exact sequence*

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

   where each P_i is a projective object in $\mathscr{C}$.

2. *Dually, an **injective resolution** of M is an exact sequence*

$$0 \to M \to I_0 \to I_1 \to \dots$$

   where each I_i is a injective object in $\mathscr{C}$.

3. *An abelian $\mathbb{k}$-category $\mathscr{C}$ has **enough projectives** if for each object M, we can find a projective object P and a surjective morphism $P \to M \to 0$.*

4. *An abelian $\mathbb{k}$-category $\mathscr{C}$ has **enough injectives** if for each object M, we can find a injective object I and an injective morphism $0 \to M \to I$.*

We will show that the category rep$Q$ has enough projectives and injectives.

Theorem 2.3.1. *Let M be a representation in repQ.*

1. *There exists a projective resolution of M of the form*

$$0 \to P_1 \to P_0 \to M \to 0.$$

2. *There exists an injective resolution of M of the form*

$$0 \to M \to I_0 \to I_1 \to 0.$$

sketch of proof. Let's explicitly construct the projective resolution of the representation $M = (M_i, \varphi_\alpha)$. Define

$$P_1 = \oplus_{\alpha \in Q_1} d_{s(\alpha)} P(t(\alpha)) \quad P_0 = \oplus_{i \in Q_0} d_i P(i)$$

where $d_i P(i)$ stands for the direct sum of d_i copies of $P(i)$ where d_i is the dimension of M_i. Then

$$0 \to P_1 \to P_0 \to M \to 0$$

is a projective resolution of M.

Dually, we define the injective resolution as follows. Define

$$I_1 = \oplus_{\alpha \in Q_1} d_{t(\alpha)} I(s(\alpha)) \quad I_0 = \oplus_{i \in Q_0} d_i I(i).$$

Then

$$0 \to M \to I_0 \to I_1 \to 0$$

is an injective resolution of M. It remains to show that these are projective/injective resolutions. This proof is quite long and technical, and can be found in Schiffler's book. □

These resolutions are called the **standard projective (injective) resolution of** M. There are other projective/injective resolutions other than the standard ones. An immediate consequence of the existence of these projective/injective resolutions is the following:

Corollary 2.3.2. *The category repQ has enough projectives and enough injectives.*

Let's construct some standard resolutions.

Example 2.3.1. Let Q be the quiver $1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3 \xrightarrow{\gamma} 4$ and consider the representations $M = S(3) = 3$ and $N = {}^{1\,3}_{2\,4} \oplus 4$. To construct the standard projective resolution of M, we begin by finding P_0. To do this we look at the *vertices* in the quiver. Note that for the representation M, the dimension of the vector space at each vertex is given by $d_1 = 0, d_2 = 0, d_3 = 1$, and $d_4 = 0$. Therefore $P_0 = 0P(1) \oplus 0P(2) \oplus 1P(3) \oplus 0P(4) = P(3)$. To find P_1, we need to look at the *arrows* in the quiver. Actually, we only need to know the dimension of the vector space at the starting point of each arrow $d_{s(\alpha)}$. For the representation M we have, $d_{s(\alpha)} = d_1 = 0, d_{s(\beta)} = d_3 = 1$, and $d_{s(\gamma)} = d_3 = 1$. Therefore, P_1 is given by $d_1 P(t(\alpha)) \oplus d_3 P(t(\beta)) \oplus d_3 P(t(\gamma)) = 0P(2) \oplus 1P(2) \oplus 1P(4) = P(2) \oplus P(4)$. We have the standard projective resolution given by

$$0 \to P(2) \oplus P(4) \to P(3) \to S(3) \to 0.$$

We can also write this standard projective resolution as

$$0 \to 2 \oplus 4 \to \begin{smallmatrix} 3 \\ 2\,4 \end{smallmatrix} \to 3 \to 0.$$

Now let's compute the standard projective resolution for N, the representation given by

$$\Bbbk \xrightarrow{1} \Bbbk \xleftarrow{1} \Bbbk \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \Bbbk^2.$$

To compute P_0, we need $d_1 = 1, d_2 = 1, d_3 = 1$, and $d_4 = 2$, so $P_0 = 1P(1) \oplus 1P(2) \oplus 1P(3) \oplus 2P(4) = P(1) \oplus P(2) \oplus P(3) \oplus P(4) \oplus P(4)$. To compute P_1 we need $d_{s(\alpha)} = d_1 = 1, d_{s(\beta)} = d_3 = 1$, and $d_{s(\gamma)} = d_3 = 1$, so $P_1 = 1P(2) \oplus 1P(2) \oplus 1P(4)$. Therefore the standard projective resolution of N is

$$0 \to P(2) \oplus P(2) \oplus P(4) \to P(1) \oplus P(2) \oplus P(3) \oplus P(4) \oplus P(4) \to N \to 0.$$

Again, another way to write this is

$$0 \to 2 \oplus 2 \oplus 4 \to \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix}3\\24\end{smallmatrix} \oplus 4 \oplus 4 \to \begin{smallmatrix}1\,3\\2\,4\end{smallmatrix} \oplus 4 \to 0. \qquad (*)$$

Notice that projective resolution $(*)$ is not 'minimal' in the sense that we can remove a copy of $P(2)$ and $P(4)$ from both P_1 and P_0 and still get a projective resolution:

$$0 \to 2 \to \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\24\end{smallmatrix} \oplus 4 \to \begin{smallmatrix}1\,3\\2\,4\end{smallmatrix} \oplus 4 \to 0.$$

Now we have constructed the two 'minimal' standard projective resolutions of M and N.

Exercise 2.3.1. Compute the 'minimal' standard injective resolutions of M and N.

But what does minimal mean anyway?

Zooming Out 11. *Let $\mathscr{C}$ be an abelian $\Bbbk$-category and $M \in Ob(\mathscr{C})$.*

1. *A **projective cover** of M is a projective object P and a surjective morphism $g : P \to M$ such that whenever $g' : P' \to M$ is a surjective morphism with P' projective, there exists a surjective morphism $h : P' \twoheadrightarrow P$ such that the following diagram commutes:*

$$\begin{tikzcd}
& P' \arrow[dl, dotted, two heads, "h"'] \arrow[d, "g'"] & \\
P \arrow[r, "g"'] & M \arrow[r] \arrow[d] & 0 \\
& 0 &
\end{tikzcd}$$

That is, a projective cover is the 'smallest' projective object that surjects onto M.

2. *An **injective envelope** of M is an injective object I and an injective morphism $f : M \to I$ such that whenever $f' : M \hookrightarrow I'$ is an injective morphism with I' injective, there exists an injective morphism $h : I \hookrightarrow I'$ such that the following diagram commutes:*

$$\begin{tikzcd}
& 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "f'"'] & I \arrow[dl, dotted, hook, "h"] \\
& I' &
\end{tikzcd}$$

That is, an injective cover is the 'smallest' injective object into which M injects.

3. *A projective resolution*

$$\cdots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

*is called **minimal** if $f_0 : P_0 \to M$ is a projective cover and $f_i : P_i \to ker(f_{i-1})$ is a projective cover for every $i \geq 1$.*

4. *An, injective resolution*
$$0 \to M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} \dots$$
is called ***minimal*** *if $f_0 : M \to I_0$ is an injective envelope and $f_i : coker(f_{i-1}) \to I_i$ is an injective envelope for every $i \geq 1$.*

Remark 2.3.1. Frankly, these definitions are technical and we don't *really* need to use them that often in this class. We just need to know that we can always get minimal projective and injective resolutions in rep$Q$ and moreover, that they are unique up to isomorphism. A proof of this for rep$Q$ can be found in Schiffler's book.

One nice thing about rep$Q$, only because our quiver does not have any cycles, is that subrepresentations of projective representations are projective! An abelian $\Bbbk$-category in which subobjects of projective objects are projective is also called **heredity**.

Theorem 2.3.3. *The category repQ is hereditary; that is, subrepresentations of projective representations are projective.*

This is nice because for instance, in the definition of minimal projective resolution, we need that the map $f_i : P_i \to \ker(f_{i-1})$ is a projective cover. But here, the kernel of f_{i-1} is a projective representation in rep$Q$ since it is a subrepresentation of the projective representation P_{i-1}. So to find a projective resolution of M, we take a projective cover $P \xrightarrow{f} M$ and get a projective resolution

$$0 \to \ker f \xrightarrow{i} P \xrightarrow{f} M \to 0,$$

which is exactly what we did in Example 2.3.1 and precisely *how* the standard projective resolution is defined! Another benefit of this is that we can order the projectives!

Corollary 2.3.4. *Let $f : M \to P$ be a nonzero morphism from an indecomposable representation M to a projective representation P. Then M is projective, and f is injective.*

Proof. Since the image of f is a subrepresentation of P, it is a projective representation by Theorem 2.3.1. Therefore the following short exact sequence ends at a projective representation, hence splits by Proposition 2.2.3:

$$0 \to \ker(f) \to M \to \operatorname{im}(f) \to 0.$$

Therefore, $M \cong \operatorname{im}(f) \oplus \ker(f)$ by Proposition 1.5.2. But we assume that M is indecomposable! So either $M \cong \operatorname{im}(f)$ and $\ker(f) = 0$ or vice versa. Since we assume that f is nonzero, $\operatorname{im}(f) \neq 0$, so we must have that $M \cong \operatorname{im}(f)$ and $\ker(f) = 0$. Therefore M is projective and f has trivial kernel, making it injective. $\square$

This is a *very* useful result! This means that we can put a 'partial order' on the projective representations of Q by inclusion. If the projective P embeds into the projective P', then P is smaller than P'.

Example 2.3.2. Let Q be the quiver $1 \to 2 \leftarrow 3$. Then we have one projective representation at each vertex, namely $P(1) = \begin{smallmatrix}1\\2\end{smallmatrix}, P(2) = S(2) = 2$, and $P(3) = \begin{smallmatrix}3\\2\end{smallmatrix}$. Recall in Example 1.5.3, we said that the subrepresentations are on the bottom and the quotient representations are on the top. That means that we have injections $P(2) \hookrightarrow P(1)$ and $P(2) \hookrightarrow P(3)$. Therefore $P(2)$ is smaller than both $P(1)$ and $P(2)$. We can actually make them into the vertices of a quiver where we write the smaller projectives on the left:

$$\begin{tikzcd}
 & \begin{smallmatrix}1\\2\end{smallmatrix} \\
2 \arrow[ur] \arrow[dr] & \\
 & \begin{smallmatrix}3\\2\end{smallmatrix}
\end{tikzcd}$$

This is the beginning of the so-called ‘Auslander-Reiten’ quiver for Q. This is a quiver that contains all the indecomposable representations as vertices and ‘irreducible’ maps between them as arrows. We will see how to fit all the other indecomposable representations into the AR quiver in chapter three. But for now we can say that when Q doesn’t have any cycles, we always start with the projectives on the left like in this example.

We end this section by finally showing that the $P(i)$ form a complete set of indecomposable representations in rep$Q$. To do this, we must introduce some new terminology.

Definition 2.3.1. Let $A = \oplus_{i \in Q_0} P(i)$. A representation $F \in \text{rep}Q$ is called **free** if $F \cong A \oplus A \oplus \cdots \oplus A$.

So free representations are direct sums of direct sums of $P(i)$.

Proposition 2.3.5. *A representation M in repQ is projective if and only if there exists a free representation F in repQ such that M is isomorphic to a direct summand of F.*

Proof. Suppose that M is a projective representation with dimension vector $\mathbf{dim}(M) = (d_1, d_2, \ldots, d_n)$ (Definition 1.2.1). Then by taking the standard projective resolution of M, we get a surjection $g : \oplus_{i \in Q_0} d_i P(i) \to M$, which gives a short exact sequence

$$0 \to \ker(g) \to \oplus d_i P(i) \to M \to 0.$$

Again by Proposition 2.2.3, since we assume that M is projective, this sequence splits and $\oplus d_i P(i) \cong \ker(g) \oplus M$ by Proposition 1.5.2. Therefore, M is a direct sum of $\oplus d_i P(i)$, which is a free representation by definition.

Conversely, suppose that M is isomorphic to a direct summand of a free representation F. Then M is a direct sum of representations of the form $P(i)$, which are projective. Since the sum of projective representations is a projective representation by Proposition 2.2.4, we have that M is projective. $\square$

Corollary 2.3.6. *Any projective representation $P \in repQ$ is a direct sum of $P(i)$'s:*

$$P \cong P(i_1) \oplus P(i_2) \oplus \cdots \oplus P(i_k)$$

with the i_j's not necessarily distinct.

Proof. By the previous proposition, any projective P is a direct summand of a free representation F, which is by definition a direct sum of direct sums of $P(i)$. □

Okay great! Now we have a complete list of indecomposable simples, $S(i)$, and indecomposable projectives, $P(i)$. We get the fact that $I(i)$ is a complete list of indecomposable injective representations from the 'Nakayama functor', which we will define in the next section.

2.4 Auslander-Reiten Translation

In this section, we will answer the question:

Question 2.4. What is Auslander-Reiten theory?

The Auslander-Reiten translate is the main tool used by mathematicians today to study the category rep$Q$. What's even cooler is that Maurice Auslander was a mathematician at Brandeis! In fact, Idun Reiten visited Brandeis and had an office on the second floor of Goldsmith. Kiyoshi Igusa, current professor at Brandeis and my advisor, told me that Maurice and Idun were frequently discussing Auslander-Reiten theory in the office next to his. Of course at the time it wasn't called that, but the AR translate is so important that today it is named after them. To define the translate, we will need some more category theory. The same way we have the notion of two vector spaces, or representations, or algebras of being 'the same' (isomorphic), there is a notion of two categories being the same.

Zooming Out 12. *Let $\mathscr{C}$ and $\mathscr{D}$ be two categories.*

1. *We say that two functors $F, G : \mathscr{C} \to \mathscr{D}$ are **functorially isomorphic** and write $F \cong G$, if for every object $M \in Ob(\mathscr{C})$, there is an isomorphism $\varphi_M : F(M) \to G(M)$ in $\mathscr{D}$ such that for every morphism $f : M \to N$ in $\mathscr{C}$, we have the following commutative diagram:*

$$\begin{tikzcd} F(M) \arrow[r, "F(f)"] \arrow[d, "\varphi_M"'] & F(N) \arrow[d, "\varphi_N"] \\ G(M) \arrow[r, "G(f)"] & G(N) \end{tikzcd}$$

2. *A covariant functor $F : \mathscr{C} \to \mathscr{D}$ is called an **equivalence of categories** if there exists a functor $G : \mathscr{D} \to \mathscr{C}$ such that $G \circ F \cong 1_{\mathscr{C}}$ and $F \circ G \cong 1_{\mathscr{D}}$. The functor G is called a **quasi-inverse** of F.*

3. *We say two categories $\mathscr{C}$ and $\mathscr{D}$ are **equivalent**, denoted by $\mathscr{C} \cong \mathscr{D}$, if there exists an equivalence of categories between them: $F : \mathscr{C} \to \mathscr{D}$.*

*4. A contravariant functor F that has a (contravariant) quasi-inverse is called a **duality**.*

*5. We say two categories $\mathscr{C}$ and $\mathscr{D}$ are **dual** if there exsists a duality between them.*

Aha! This is where the word dual comes from! And also, a duality is contravariant, which means it flips the arrows! Let's see an example of a duality.

Let Q be a quiver without cycles and let Q^{op} denote the **opposite quiver**; that is, the quiver obtained from Q by reversing all the arrows:

$$Q: 1 \to 2 \leftarrow 3 \qquad Q^{op}: 1 \leftarrow 2 \to 3.$$

If $Q = (Q_0, Q_1, s, t)$, then $Q^{op} = (Q_0, Q_1^{op})$ where

$$Q_1^{op} = \{\alpha^{op} : s(\alpha^{op}) = t(\alpha),\ \text{and } t(\alpha^{op}) = s(\alpha)\}.$$

Example 2.4.1. A **duality** between rep$Q$ and repQ^{op} is given by

$$D = \mathrm{Hom}_{\Bbbk}(-, \Bbbk) : \mathrm{rep}Q \to \mathrm{rep}Q^{op}.$$

When we refer to *the duality*, this is what we mean. The duality D is a contravariant Hom functor that is a little different than the one we've seen in the past. Instead of looking at all morphisms into a *representation* of Q, we're looking at all the $\Bbbk$-linear maps from a morphism *into the field* $\Bbbk$. Namely, on representations $M = (M_i, \varphi_\alpha)$, we define

$$DM = (DM_i, D\varphi_{\alpha^{op}}),$$

which is an object in repQ^{op} by definition, where DM_i is the **dual vector space** of M_i; that is, $DM_i := \mathrm{Hom}_{\Bbbk}(M_i, \Bbbk)$. So the dual vector space is the vector space of linear transformations from M_i to $\Bbbk$. The dual of a vector space is isomorphic to the original vector space in a 'canonical way'; that is, there is only *one* isomorphism between them up to 'equivalence'. Therefore, the dual space is algebraically really just the same vector space as the original one.

Exercise 2.4.1. Show that $DV \cong V$ for V a finite dimensional vector space over a field $\Bbbk$.

Now that we know how to define the vector spaces, we need to define the linear transformations $\varphi_{\alpha^{op}}$, which we define to be the *pullback* of φ_α:

$$D\varphi_{\alpha^{op}} : DM_{t(\alpha)} \to DM_{s(\alpha)} \qquad u \mapsto u \circ \varphi_\alpha$$

Let's check that this makes sense. The linear transformation $\varphi_{\alpha^{op}}$ is defined on $\mathrm{Hom}_{\Bbbk}(M_{t(\alpha)}, \Bbbk)$, so it eats morphisms u from $M_{t(\alpha)}$ to $\Bbbk$, and returns morphisms from $M_{s(\alpha)}$ to $\Bbbk$. We also have a linear transformation $\varphi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$. So by doing φ_α first and then doing u, we have a morphism

$$u \circ \varphi_\alpha : M_{s(\alpha)} \xrightarrow{\varphi_\alpha} M_{t(\alpha)} \xrightarrow{u} \Bbbk \in \mathrm{Hom}(M_{s(\alpha)}, \Bbbk),$$

which is precisely what we need. Here is a picture to see how this is *pulling back* φ_α:

$$\begin{tikzcd} M_{s(\alpha)} \arrow[r, "\varphi_\alpha"] \arrow[dr, "u\circ\varphi_\alpha"'] & M_{t(\alpha)} \arrow[d, "u"] \\ & \Bbbk \end{tikzcd}$$

Okay great, we have defined the duality on objects $M \in \mathrm{Ob}(\mathrm{rep}Q)$; namely, $M \mapsto DM$ defined above. Now we need to define it on morphisms! So let $f : M \to N$ be a morphism of objects in rep$Q$. Since the duality is contravariant, the arrows flip! So we need to define $Df : DN \to DM$ in repQ^{op}. Let $u = (u_1, u_2, \dots, u_n) \in DN$. Then each $u_i : N_i \to \Bbbk$ is a linear transformation from N_i to $\Bbbk$ by definition. Define $Df(u) = u \circ f$. Then at each vertex in the quiver i, we have

$$u_i \circ f_i : M_i \xrightarrow{f_i} N_i \xrightarrow{u_i} \Bbbk$$

is a linear transformation from M_i to $\Bbbk$, which is precisely what we need for $Df(u) \in DM$. Moreover, since the composition of morphisms of representations is a morphism of representations, $Df : DN \to DM$ is well defined. Here is a picture:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "u\circ f"'] & N \arrow[d, "u"] \\ & \Bbbk \end{tikzcd}$$

Now we have defined the duality on both objects and morphisms! But to make it a duality, we need a quasi-inverse. It turns out that this functor is not too difficult to find. We can define the exact same functor

$$D^{op} = \mathrm{Hom}_{\Bbbk}(-, \Bbbk) = D : \mathrm{rep}Q^{op} \to \mathrm{rep}Q$$

in an analogous way!

Exercise 2.4.2. Write down a complete definition of D^{op}. Show that $D \circ D^{op} \cong 1_{\mathrm{rep}Q^{op}}$ and $D^{op} \circ D \cong 1_{\mathrm{rep}Q}$. There is a lot to do in this exercise, so make sure to talk with your classmates (and me) about this one!

Now let's see what the duality does to projectives. Let $P_Q(i)$ denote the projective representation of Q at vertex i. Then $P_Q(i)_j$ has a basis consisting of all paths from i to j. After we apply the duality, $DP_Q(i)_j = \mathrm{Hom}(P_Q(i)_j, \Bbbk) \cong P_Q(i)_j$, so the dimension of the vector space does not change after applying the duality. What *does* change though is that all the arrows in the quiver Q are flipped! This means that paths from i to j in Q are now paths from j to i in Q^{op}. That is, $DP_Q(i)_j$ has a basis consisting of all paths from j to i in Q^{op}. But we have a name for this! This is precisely the vector space at vertex j of the *injective* representation at vertex i of the opposite quiver! In other words $DP_Q(i)_j \cong I_{Q^{op}}(i)_j$. With some more work, we have the following proposition:

Proposition 2.4.1. *We have $DP_Q(i) = I_{Q^{op}}(i)$ for all vertices $i \in Q_0$.*

Exercise 2.4.3. Prove this proposition. Use the fact that $DP_Q(i)_j = I_{Q^{op}}(i)_j$ shown in the paragraph above. Then all that is left to show is that the maps are the same.

Let's look at some consequences of this proposition. Fix the quiver Q and define proj$Q$ to be the category of projective representations of Q and the morphisms are all the morphisms between these representations in rep$Q$. Then we know by Krull-Schmidt, Theorem 1.3.1, any representation of Q decomposes as a direct sum of indecomposable representations in an essentially unique way. Now by Proposition 2.2.4 and Corollary 2.3.6, any projective representation decomposes as the direct sum of the $P(i)$'s. Thus to study proj$Q$, it suffices to study the $P(i)$'s. But by the previous proposition, Proposition 2.4.1, we know that this is equivalent to studying the injectives at vertex i over the opposite quiver! Using a similar argument, we have that studying these injectives is equivalent to studying injQ^{op}, the category of injective representations of the quiver Q^{op}. Therefore, we have the following corollary:

Corollary 2.4.2. *The duality $D : repQ \to repQ^{op}$ restricts to a duality $projQ \to injQ^{op}$.*

Let's see an explicit example of the duality at work.

Example 2.4.2. Let $Q : 1 \to 2 \leftarrow 3$. Then we have the three indecomposable projective representations:

$$P(1) = \begin{matrix} 1 \\ 2 \end{matrix} = \Bbbk \xrightarrow{1} \Bbbk \leftarrow 0$$

$$P(2) = S(2) = 2 = 0 \to \Bbbk \leftarrow 0$$

$$P(3) = \begin{matrix} 3 \\ 2 \end{matrix} = 0 \to \Bbbk \xleftarrow{1} \Bbbk$$

Applying the duality gives the following representations:

$$DP(1) = \begin{matrix} 2 \\ 1 \end{matrix} = \Bbbk \xleftarrow{1} \Bbbk \to 0$$

$$DP(2) = DS(2) = 2 = 0 \leftarrow \Bbbk \to 0$$

$$DP(3) = \begin{matrix} 2 \\ 3 \end{matrix} = 0 \leftarrow \Bbbk \xrightarrow{1} \Bbbk$$

Notice that over the opposite quiver $Q^{op} : 1 \leftarrow 2 \to 3$, these representations are precisely $I_{Q^{op}}(1), I_{Q^{op}}(2)$, and $I_{Q^{op}}(3)$ from top to bottom respectively, illustrating the result in Proposition 2.4.1.

Along with some other small details missing, we didn't give a full proof of Corollary 2.4.2. One reason for this is that we didn't technically show that the injectives at vertex i form a complete set of injectives. We need this in order to build all the injective representations in inj$Q$. To show this, we will use the very important and powerful Nakayama functor, which we define now as a composition of the duality with another contravariant Hom functor.

Recall the notion of the contravariant Hom functor, $\mathrm{Hom}(-, Y)$, defined just after Zooming Out 8. This was a functor from rep$Q$ to $\mathbf{Vect}_{\Bbbk}$. It turns out that when we make a special choice of Y, we can actually make this a functor from rep$Q$ to repQ^{op}. In particular, consider the free Q-representation $A = \oplus P_Q(i)$. Then for any representation X in rep$Q$, the vector space $\mathrm{Hom}(X, A)$ is actually a representation! This does *not* happen in general and relies on the structure of the free representation A. Let's denote the representation by $M = \mathrm{Hom}(X, A)$. To define this Q^{op} representation, we must first define the vectorspaces, whence define $M_i := \mathrm{Hom}(X, P(i))$.

Now let α be an arrow from i to j in Q_0. By Corollary 2.2.8, $\mathrm{Hom}(P(j), P(i))$ has basis consisting of all paths from i to j. Thus α is an element in $\mathrm{Hom}(P(j), P(i))$; that is, it defines a map from $P(j) \to P(i)$ by taking any element $p_j \in P(j)$ and assigning to it the element $\alpha p_j \in P(i)$. To define the linear transformations in the representation M, ie $\varphi_{\alpha^{op}} : \mathrm{Hom}(X, P(j)) \to \mathrm{Hom}(X, P(i))$, let $f \in \mathrm{Hom}(X, P(j))$. We define $\varphi_{\alpha^{op}}(f) = \alpha \circ f \in \mathrm{Hom}(X, P(i))$ as in the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\varphi_{\alpha^{op}}(f)"'] & P(j) \arrow[d, "\alpha"] \\ & P(i) \end{tikzcd}$$

Great, we have given $\mathrm{Hom}(X, A) = (M_i, \varphi_{\alpha^{op}})$ the structure of a Q^{op} representation, but remember, we wish to make $\mathrm{Hom}(-, A)$ a functor to repQ^{op}. To do this, we must also define it on morphisms, so let $g : M \to N$ be a morphism in rep$Q$. We wish to define a morphism $\mathrm{Hom}(g, A) = g^* : \mathrm{Hom}(N, A) \to \mathrm{Hom}(M, A)$ in repQ^{op}. To do this, we take the pullback of g as the notation may suggest; that is, $g^*(f) = f \circ g \in \mathrm{Hom}(M, A)$. To make sure that g^* is a morphism in repQ^{op}, we need to make sure the following box commutes:

$$\begin{tikzcd} \mathrm{Hom}(N, P(i)) \arrow[d, "g_i^*"'] & \mathrm{Hom}(N, P(j)) \arrow[l, "\varphi_{\alpha^{op}}"'] \arrow[d, "g_j^*"] \\ \mathrm{Hom}(M, P(i)) & \mathrm{Hom}(M, P(j)) \arrow[l, "\psi_{\alpha^{op}}"] \end{tikzcd}$$

We start in the top right corner by letting $f : N \to P(j) \in \mathrm{Hom}(N, P(j))$ be a morphism of representations. Then

$$\begin{aligned} g_i^*(\varphi_{\alpha^{op}}(f)) &= g_i^*(\alpha \circ f) && \text{(by definition of } \varphi_{\alpha^{op}}) \\ &= (\alpha \circ f) \circ g && \text{(by definition of } g^*) \\ &= \alpha \circ (f \circ g) && \text{(Associativity of composition)} \\ &= \psi_{\alpha^{op}}(f \circ g) && \text{(by definition of } \psi_{\alpha^{op}}) \\ &= \psi_{\alpha^{op}}(g_j^*(f)) && \text{(by definition of } g^*) \end{aligned}$$

Therefore the box commutes and we have proven:

Lemma 2.4.3. *The functor Hom$(-, A)$ is one from repQ to repQ^{op}.*

We are now ready to define the Nakayama functor!

Definition 2.4.1. The functor $\nu = \mathrm{DHom}(-, A)$ is called the Nakayama functor.

A functor from a category to itself is called an **endofunctor**. So the Nakayama functor is an endofunctor from rep$Q$ to itself, defined as the composition of $\mathrm{Hom}(-, A)$ and the duality D:

$$\begin{tikzcd} \mathrm{rep}Q \arrow[r, "\mathrm{Hom}(-,A)"] \arrow[rr, bend right, "\nu"'] & \mathrm{rep}Q^{op} \arrow[r, "D"] & \mathrm{rep}Q \end{tikzcd}$$

We wish to use the Nakayama functor to help us describe the indecomposable representations of Q. Recall Corollary 2.3.4. This Corollary states that there are no non-zero maps into projective representations unless they are coming from projective representations; that is, $\mathrm{Hom}(M, P) = 0$ for any projective representation P and any non-projective representation M. Since $A = \oplus P(i)$ is a projective representation, this holds for $\mathrm{Hom}(-, A)$, and hence the Nakayama functor. Thus to see how we can use the Nakayama functor to study the indecomposable representations, we can simply look at what it does to indecomposable projective representations. Let's compute $\nu(P_Q(i)) = \mathrm{DHom}(P_Q(i), A)$. We first need to compute the representation $M = \mathrm{Hom}(P_Q(i), A) = (\mathrm{Hom}(P_Q(i), P_Q(j)), \varphi_{\alpha^{op}})$. The vector space at vertex j is given by $M_j = \mathrm{Hom}(P_Q(i), P_Q(j))$, which by Corollary 2.2.8, is the vector space with basis consisting of all paths from j to i in Q. This is equivalent to being the vector space in Q^{op} whose basis consists of all paths from i to j since we flip the arrows in Q^{op}. This is the same thing as the vector space $P_{Q^{op}}(i)_j$.

Moreover, given any arrow $\alpha : h \to j \in Q_1$, the morphism $\varphi_{\alpha^{op}} : P_{Q^{op}}(i)_j \to P_{Q^{op}}(i)_h$ is defined by $p \mapsto \alpha p$. This is *precisely* the definition of the morphisms in the projective representation $P_{Q^{op}}(i)$, and therefore we conclude that $\mathrm{Hom}(P_Q(i), A) = P_{Q^{op}}(i)$, the indecomposable projective Q^{op} representation at vertex i. To see where the Nakayama functor sends $P_Q(i)$, we must now find the dual of $\mathrm{Hom}(P_Q(i), A) = P_{Q^{op}}(i)$. But by Proposition 2.4.1, we know that the dual of a projective at vertex i is just the injective at vertex i over the opposite quiver! Thus, $D(P_{Q^{op}}(i)) = I_Q(i) = \nu(P_Q(i))$.

Wait! Why is the opposite of the opposite the original quiver?

Therefore, we have shown that the Nakayama functor maps the indecomposable projectives to indecomposable injectives. Actually, even more is true:

Theorem 2.4.4. *The restriction of the Nakayama functor to projQ gives an equivalence of categories $\nu : projQ \to injQ$ whose quasi-inverse is given by*

$$\nu^{-1} : Hom(DA^{op}, -) : injQ \to projQ,$$

where $DA^{op} = \oplus I_Q(i)$. Moreover, for any vertex i,

$$\nu(P(i)) = I(i) \text{ and } \nu^{-1}(I(i)) = P(i),$$

and given a path c from i to j, which defines a linear transformation $f_c : P(j) \to P(j)$, we have

$$\nu(f_c) : I(j) \to I(i)$$

is given by deleting the path c.

Although we will not prove this, hopefully there is enough evidence to make this seem believable to the reader. A proof can be found in Schiffler's book. The key take away here is that proj$Q \cong$ inj$Q$, which means that the isoclasses of indecomposable injectives are precisely the representations that the isoclasses of indecomposable projectives are sent to by the Nakayama functor! In particular, this along with Corollary 2.3.6, allow us to finally conclude:

Corollary 2.4.5. *Any injective representation $I \in repQ$ is a direct sum of $I(i)$'s:*

$$I \cong I(i_1) \oplus I(i_2) \oplus \cdots \oplus I(i_k)$$

with the i_j's not necessarily distinct.

Let's see an example of the Nakayama functor at work:

Example 2.4.3. Let Q be the quiver from Example 2.3.2, so $Q = 1 \to 2 \leftarrow 3$ and we have one indecomposable projective representation at each vertex, namely $P(1) = {\substack{1\\2}}$, $P(2) = S(2) = 2$, and $P(3) = {\substack{3\\2}}$. The injectives are given by $\nu(P(1)) = I(1) = 1$, $\nu(P(2)) = I(2) = {\substack{3\,1\\2}}$, and $\nu(P(3)) = I(3) = 3$. We can also partially order the injectives a similar way in which we ordered the projectives. Namely, an injective I is smaller than an injective I' if there is a surjective morphism from I' to I. This follows from a dual statement to Corollary 2.3.4. Below are the two orderings realized as quivers:

$$\begin{tikzcd}
 & P(1) & & & I(1) \\
P(2) \arrow[ur] \arrow[dr] & & & I(2) \arrow[ur] \arrow[dr] & \\
 & P(3) & & & I(3)
\end{tikzcd}$$

Notice that the Nakayama functor moves proj$Q$, the quiver on the left, to inj$Q$, the quvier on the right. We will see this again in the next chapter.

We need one more result about the Nakayama functor before being able to define the Auslander-Reiten translates.

Proposition 2.4.6. *Given an exact sequence in repQ of the form $L \xrightarrow{f} M \xrightarrow{g} N \to 0$, we have that*

$$\nu(L) \xrightarrow{\nu(f)} \nu(M) \xrightarrow{\nu(g)} \nu(N) \to 0$$

is also an exact sequence in repQ.

We have seen something like this before. In Theorems 1.6.1 and 1.6.3 we have seen that the Hom functor satisfies some very similar properties. Actually, these ideas are categorical!

Zooming Out 13. *Let $\mathscr{C}$ and $\mathscr{D}$ be two abelian $\mathbb{k}$-categories. A covariant (or contravariant) functor $F : \mathscr{C} \to \mathscr{D}$ is called* ***exact*** *if it maps exact sequences in $\mathscr{C}$ to exact sequences in $\mathscr{D}$. Any duality or equivalence is exact for instance. This property is a bit too strong though for most functors. For instance we've seen that the Hom functor maps exact sequences to sequences that are not quite exact but pretty close.*

If F is a covariant functor, then it is called ***left exact*** *if for any exact sequence*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N$$

the sequence

$$0 \to F(L) \stackrel{F(f)}{\to} F(M) \stackrel{F(g)}{\to} F(N)$$

*is exact. The functor F is called **right exact** if for any sequence*

$$L \stackrel{f}{\to} M \stackrel{g}{\to} N \to 0$$

the sequence

$$F(L) \stackrel{F(f)}{\to} F(M) \stackrel{F(g)}{\to} F(N) \to 0$$

is exact.

*If $G : \mathscr{C} \to \mathscr{D}$ is a contravariant functor, then G is called **left exact** if for any exact sequence*

$$L \stackrel{f}{\to} M \stackrel{g}{\to} N \to 0$$

the sequence

$$0 \to G(N) \stackrel{G(g)}{\to} G(M) \stackrel{G(f)}{\to} G(L)$$

*is exact. The functor G is called **right exact** if for any exact sequence*

$$0 \to L \stackrel{f}{\to} M \stackrel{g}{\to} N$$

the sequence

$$G(N) \stackrel{G(g)}{\to} G(M) \stackrel{G(f)}{\to} G(L) \to 0$$

is exact.

The details of the proof of Proposition 2.4.6 are left as an exercise; however, the proof follows from the fact that the duality D is exact since it is an equivalence of categories, and the functor $\mathrm{Hom}(-, X)$ is left exact as we have seen in Theorem 1.6.3. Therefore the composition of them both is right exact.

Exercise 2.4.4. Prove Proposition 2.4.6 in detail.

Okay, we are now ready to define the Auslander-Reiten Translates! Recall that Q is a quiver without oriented cycles and let M be an indecomposable representation of Q.

Definition 2.4.2. Let

$$0 \to P_1 \stackrel{p_1}{\to} P_0 \stackrel{p_0}{\to} M \to 0$$

be a minimal projective resolution of M. Applying the right exact Nakayama functor, by Proposition 2.4.6, we get an exact sequence

$$\nu(P_1) \stackrel{\nu(p_1)}{\to} \nu(P_0) \stackrel{\nu(p_0)}{\to} \nu(M) \to 0.$$

We can make this a short exact sequence by looking at the kernel of $\nu(p_1)$ as follows:

$$0 \to \ker(\nu(p_1)) \to \nu(P_1) \stackrel{\nu(p_1)}{\to} \nu(P_0) \stackrel{\nu(p_0)}{\to} \nu(M) \to 0.$$

We define the **Auslander-Reiten translate of** M, denoted by τM, by $\tau M := \ker(\nu(p_1))$. Therefore, we get an exact sequence

$$0 \to \tau M \to \nu(P_1) \stackrel{\nu(p_1)}{\to} \nu(P_0) \stackrel{\nu(p_0)}{\to} \nu(M) \to 0.$$

Definition 2.4.3. Dually, let

$$0 \to M \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \to 0$$

be a minimal injective resolution of M. Applying the left exact inverse Nakayama functor, we get an exact sequence that we can extend to the right by taking the cokernel:

$$0 \to \nu^{-1}(M) \xrightarrow{\nu^{-1}(i_0)} \nu^{-1}(I_0) \xrightarrow{\nu^{-1}(i_1)} \nu^{-1}(I_1) \to \operatorname{coker}(\nu^{-1}(i_1)) \to 0.$$

We define the **inverse Auslander-Reiten translate of** M, denoted by $\tau^{-1}M$, by $\tau^{-1}M := \operatorname{coker}(\nu^{-1}(i_1))$. Therefore, we get an exact sequence

$$0 \to \nu^{-1}(M) \xrightarrow{\nu^{-1}(i_0)} \nu^{-1}(I_0) \xrightarrow{\nu^{-1}(i_1)} \nu^{-1}(I_1) \to \tau^{-1}M \to 0.$$

Let's compute some AR-translates.

Example 2.4.4. Recall Example 2.3.1. We had $Q = 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3 \xrightarrow{\gamma} 4$, and $M = S(3) = 3$. We found a minimal projective resolution of M given by

$$0 \to P(2) \oplus P(4) \to P(3) \to S(3) \to 0.$$

We can also write this standard projective resolution as

$$0 \to 2 \oplus 4 \to \begin{smallmatrix} 3 \\ 2\,4 \end{smallmatrix} \to 3 \to 0.$$

To find τM, we need to apply the Nakayama functor to the minimal projective resolution to get

$$0 \to \tau M \to I(2) \oplus I(4) \to I(3) \to \nu S(3) \to 0,$$

which can also be written

$$0 \to \tau M \to \begin{smallmatrix} 13 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \to 3 \to 0.$$

Notice that $\nu(S(3)) = 0$ since $S(3)$ is not projective. Then τM is the kernel of the map $\begin{smallmatrix} 13 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \to 3$. The stuff on the left hand side that gets sent to zero is a 1 and 3 on the top along with the 2 and 4 on the bottom. Thus the kernel is precisely the representation $\begin{smallmatrix} 1\,3 \\ 2\,4 \end{smallmatrix} = \tau M$.

Exercise 2.4.5. Compute $\tau^{-1}M, \tau N$, and $\tau^{-1}N$ for M and N from Example 2.3.1.

Exercise 2.4.6.

1. Let P be an indecomposable projective representation of Q. Show that $\tau P = 0$.

2. Let I be an indecomposable injective representation of Q. Show that $\tau^{-1} I = 0$.

3 Auslander-Reiten Quivers

The main tool used in achieving our semester long goal is called the Auslander-Reiten quiver. In general, this is a *very* good first attempt at understanding rep$Q$. In the case when there are only finitely many indecomposable representations of Q, the Auslander-Reiten quiver actually tells us everything about rep$Q$. Recall that we have the Krull-Schmidt theorem, Theorem 1.3.1. This theorem says that the indecomposable representations are our building blocks for our representations. Isoclasses of indecomposable representations form the vertices in the Auslander-Reiten quiver, so the vertices are the building blocks of the representations.

The arrows are given by so-called *irreducible* morphisms between representations. Intuitively, these are morphisms that don't factor through another representation, so irreducible morphisms are usually between representations that are very close (but not the same) in dimension. Often times, all morphisms are given by composing irreducible morphisms, though this is not always true. In the cases we will discuss in this chapter though, this will be true. So in a certain sense, the morphisms between representations are built out of the irreducible morphisms and we can think of our arrows as the building blocks for our morphisms.

Finally, even though this quiver already captures so much information about the category rep$Q$, it also provides information about so-called *almost split sequences*. An *almost split sequence* in rep$Q$ is a short exact sequence of representations of the form

$$0 \to \tau M \xrightarrow{f} \oplus E_i \xrightarrow{g} M \to 0$$

where both f and g are irreducible. These almost split sequences (or Auslander-Reiten sequences) form so called *meshes* in the Auslander-Reiten quiver as in Figure 1.

1.
$$\begin{tikzcd} & E_1 \arrow[dr] & \\ \tau M \arrow[ur] & & M \end{tikzcd}$$

2.
$$\begin{tikzcd} \tau M \arrow[dr] & & M \\ & E_1 \arrow[ur] & \end{tikzcd}$$

3.
$$\begin{tikzcd} & E_1 \arrow[dr] & \\ \tau M \arrow[ur] \arrow[dr] & & M \\ & E_2 \arrow[ur] & \end{tikzcd}$$

4.
$$\begin{tikzcd} & E_1 \arrow[dr] & \\ \tau M \arrow[ur] \arrow[r] \arrow[dr] & E_2 \arrow[r] & M \\ & E_3 \arrow[ur] & \end{tikzcd}$$

Figure 1: Four Types of Meshes

In these meshes, the morphisms and the inverse AR translate τ^{-1} move left to right while the AR translate τ moves right to left. Though out of the scope of this course, it turns out that one can show that every indecomposable representation of Q sits in a mesh like this. It is called the existence theorem of almost split sequences in rep$Q$. In this chapter we will learn techniques to compute AR quivers for two different types of quiver.

3.1 Quivers of Type $\mathbb{A}$

Question 3.1. How do we construct the AR-quiver for quivers of type $\mathbb{A}$?

Before we address this question we need a definition.

Definition 3.1.1. A **quiver of type** $\mathbb{A}_n$ is one whose underlying graph is of the form

$$1 - 2 - \cdots - n.$$

This is just a graph. To make it a quiver, we must direct the arrows so they have initial and terminal points. We can orient the graph in any way we'd like. We will introduce three methods for studying the AR quiver for quivers of type $\mathbb{A}_n$.

Method 3.1.1. τ-orbits

Definition 3.1.2. Let M be an object of rep$Q$. The τ-orbit of M is the following set: $\{N \in \mathrm{Ob}(\mathrm{rep}Q) :\ \text{there exists } i \in \mathbb{Z} \text{ with } N = \tau^i M\}$

The method of τ-orbits is essentially by definition. Since the maps in the AR quiver move left to right, by Corollary 2.3.4, the projectives must be all the way on the left in the AR quiver. Then for each projective we repeatedly compute its inverse AR translate until it turns into an injective representation. At this point, by Exercise 2.4.6, we have reached the end of the AR quiver since one more iteration of τ^{-1} will be zero. For quivers of type $\mathbb{A}$, there is only one projective and one injective on each τ-orbit. Thus, this method of construction produces all the possible meshes and hence the entire AR quiver piece by piece through making horizontal strips. Let's do an example.

Example 3.1.1. Let $Q = 1 \to 2$. This is a quiver of type $\mathbb{A}_2$. We know that there are two projectives, $P(1) = \begin{smallmatrix}1\\2\end{smallmatrix}$ and $P(2) = 2$ and that they form the left boundary of the AR quiver:

$$\begin{tikzcd} & P(1) \\ P(2) \arrow[ur] & \end{tikzcd}$$

Notice that this is the beginning of a type 1 mesh from Figure 1. Now we compute the two τ^{-1} orbits (one for each projective) by first computing $\tau^{-1}P(2)$. To do this, we must take a minimal injective resolution of $P(2)$. By the proof of Theorem 2.3.1, this is given by

$$0 \to 2 \to I(2) \to I(1) \to 0.$$

Now we apply the inverse Nakayama functor to this minimal injective resolution to attain

$$0 \to \nu^{-1}2 \to P(2) \to P(1).$$

Since $P(2) = 2$ and $P(1) = \begin{smallmatrix}1\\2\end{smallmatrix}$, by definition, $\tau^{-1}P(2)$ is the cokernel of the map $2 \to \begin{smallmatrix}1\\2\end{smallmatrix}$. This is isomorphic to $S(1) = 1$; therefore, we have that $\tau^{-1}P(2) = S(1)$ and we have completed a mesh! The next step in the AR-quiver is the following:

$$
\begin{tikzcd}
 & P(1) \arrow[dr] & \\
P(2) \arrow[ur] & & S(1)
\end{tikzcd}
$$

Since $S(1) = I(1)$ is an injective representation, we are done with this τ^{-1} orbit. So now we do the same thing for $P(1)$. But wait! The representation $P(1) = I(2)$ is both projective and injective! Thus $\tau^{-1}P(1) = 0$ and we have computed our first AR quiver!

Exercise 3.1.1. Write the AR quiver for the quiver $1 \rightarrow 2 \leftarrow 3$

Method 3.1.2. The Knitting Algorithm

The method of τ-orbits can be tedious since it involves several computations. The knitting algorithm however, is a much faster method of constructing AR-quivers of type $\mathbb{A}_n$. Intuitively, it comes down to completing meshes by 'knitting' them together one at a time. This recursive process, as we will see, builds the next *vertical strip* in the AR-quiver, where the method of computing τ-orbits constructs the AR-quiver by making horizontal strips. There are three types of meshes in the AR quiver of type $\mathbb{A}_n$, namely type 1, 2, and 3 in Figure 1. Pictorially how we knit the meshes can be seen in Figure 2.

(i)

$$
\begin{tikzcd}
 & E_1 & & & E_1 \arrow[dr] & \\
M \arrow[ur] \arrow[dr] & & \rightsquigarrow & M \arrow[ur] \arrow[dr] & & \tau^{-1}M \\
 & E_2 & & & E_2 \arrow[ur] &
\end{tikzcd}
$$

(ii)

$$
\begin{tikzcd}
 & E_1 & & & E_1 \arrow[dr] & \\
M \arrow[ur] & & \rightsquigarrow & M \arrow[ur] & & \tau^{-1}M
\end{tikzcd}
$$

(iii)

$$
\begin{tikzcd}
M \arrow[dr] & & \rightsquigarrow & M \arrow[dr] & & \tau^{-1}M \\
 & E_1 & & & E_1 \arrow[ur] &
\end{tikzcd}
$$

Figure 2: The Three Types of Knitting in Type $\mathbb{A}$

As with any game, there are rules we need to follow to knit these meshes!

Definition 3.1.3. (Knitting Algorithm)

1. Compute the indecomposable projectives $P(1), P(2), \ldots, P(n)$

2. Draw an arrow from $P(i)$ to $P(j)$ whenever there is an arrow $j \xrightarrow{\alpha} i$ in Q such that each $P(i)$ sits at a different level in the AR quiver.

3. Determine which type of mesh is incomplete ((i), (ii), or (iii) in Figure 2), and knit the mesh such that
$$\mathbf{dim}\tau^{-1}M = \sum_i \mathbf{dim}E_i - \mathbf{dim}M.$$

4. Repeat the previous step until we get negative integers in the dimension vector.

Let's see this at work.

Example 3.1.2. Let $Q = 1 \to 2 \leftarrow 3 \to 4$.

Step 1 Compute the indecomposable projectives:

We have the projectives: $P(1) = \begin{smallmatrix}1\\2\end{smallmatrix}, P(2) = 2, P(3) = \begin{smallmatrix}3\\24\end{smallmatrix}$, and $P(4) = 4$.

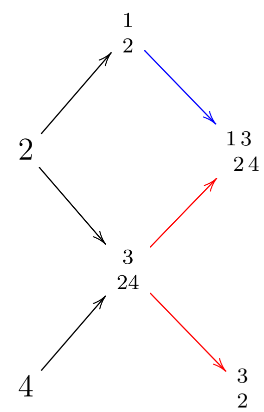
Step 2 Line up the projectives in their partial order:

We have a map from $P(2)$ into both $P(1)$ and $P(3)$. We also have a map from $P(4)$ into $P(3)$. Thus we should have arrows from $P(2)$ to both $P(1)$ and $P(3)$ and an arrow from $P(4)$ to $P(3)$. Since there are three arrows in the quiver, we have found all the arrows between projectives in the AR quiver. Here is one way to start the AR quiver:

$$\begin{tikzcd}
 & \begin{smallmatrix}1\\2\end{smallmatrix} \\
2 \arrow[ur] \arrow[dr] & \\
 & \begin{smallmatrix}3\\24\end{smallmatrix} \\
4 \arrow[ur] &
\end{tikzcd}$$

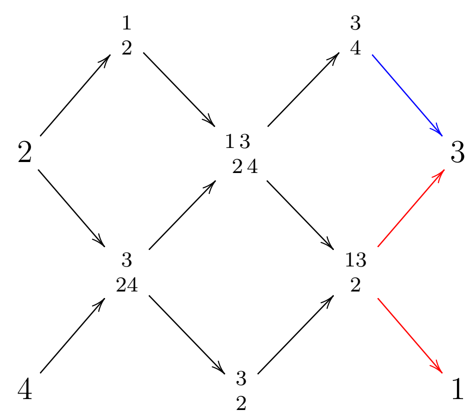
Step 3 Knit the meshes:

There are two meshes that can be knitted here: the blue one and the red one. Let's knit the red one first. We need a representation $\tau^{-1}4$ whose dimension vector is given by $\mathbf{dim}\begin{smallmatrix}3\\24\end{smallmatrix} - \mathbf{dim}4 = (0,1,1,1) - (0,0,0,1) = (0,1,1,0)$. Thus $\mathbf{dim}\tau_{-1}L = (0,1,1,0)$, so $\tau^{-1}4 = \begin{smallmatrix}3\\2\end{smallmatrix}$. To knit the blue mesh, we compute $(1,1,0,0)+(0,1,1,1)-(0,1,0,0) = (1,1,1,1)$, so $\tau^{-1}2 = \begin{smallmatrix}1\,3\\2\,4\end{smallmatrix}$. We have knitted the meshes and therefore have completed step 3:



Step 4 Repeat step 3 until done.

We continue to knit the meshes until we arrive at the following AR quiver:



To check that we are done, let's try to complete the blue mesh. We would need $\mathbf{dim}\tau^{-1}\begin{smallmatrix}3\\4\end{smallmatrix} = (0,0,1,0) - (0,0,1,1) = (0,0,0,-1)$. But no representation in rep$Q$ can have this dimension vector, so we are done. Similarly, knitting the red mesh would also produce a negative dimension vector (check it!). Therefore, we have knitted the AR quiver for Q.

Remark 3.1.1. Actually, we can extend the AR quiver to have objects in a different category called the cluster category. These objects with negative dimension vectors correspond to 'shifted' representations. But this is far beyond the scope of this course.

One idea that we have slipped under the rug here is that representations of type $\mathbb{A}$ are *uniquely determined by their dimension vector*. If there were two or more indecomposable representations that had the same dimension vector, we could not use the knitting algorithm.

Lemma 3.1.1. *Indecomposable representations of quivers of type $\mathbb{A}$ are uniquely determined up to isomorphism by their dimension vector.*

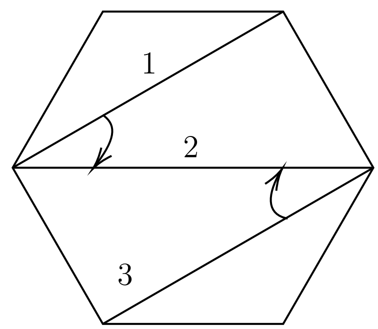
Sketch of Proof. Let $\boldsymbol{v} = (0,0,\ldots,0,1,1\ldots,1,0,0\ldots,0)$ be a dimension vector. We define the corresponding representation as $M_{\boldsymbol{v}} = (M_i, \varphi_\alpha)$ where $M_i = \Bbbk$ if the ith entry in $\boldsymbol{v}$ is 1 and $M_1 = 0$ otherwise. The map φ_α is the identity if both $M_{s(\alpha)}$ and $M_{t(\alpha)}$ are nontrivial and the zero map otherwise. This association is unique up to isomorphism. $\square$

It is precisely this lemma on which the next method relies.

Method 3.1.3. Diagonals of a Polygon with $n+3$ Vertices

Let Q be a quiver of type $\mathbb{A}_n$. Then to this quiver we can associate a triangulation of a regular $n+3$-gon. Before providing a formal definition, let's give an example.

Example 3.1.3. Let $Q = 1 \to 2 \leftarrow 3$ be the quiver from Example 2.3.2. Then this corresponds to the following triangulation of a $3+3=6$-gon.



Each line segment through the interior of the polygon is called a **diagonal**. A maximal collection of diagonals that don't cross is called a **triangulation**. In the case of regular polygons, a triangulation separates the polygon into a bunch of triangles.

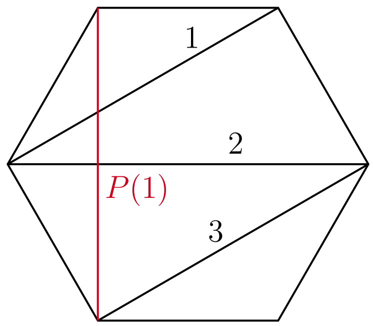
We can associate a quiver to this triangulation with one vertex for each diagonal. Now in this example, let's look at the diagonals labeled 2 and 3. If we put a small circle around the vertex at which they meet, we would need to traverse this circle clockwise to get from 3 to 2 while staying in the polygon. Therefore, 2 is clockwise from 3 and we draw an arrow from 3 to 2 in the quiver Q. Similarly, we draw an arrow from 1 to 2 in Q since 2 is also clockwise from 1. Therefore this triangulation gives us back the quiver we started with! We can also associate triangulations of $n+3$-gons to quivers of type $\mathbb{A}_n$, but not in a unique way. For the quiver in this example, to get a triangulation, we would need three non-crossing diagonals labeled 1, 2, and 3 placed in the hexagon such that 2 is clockwise from 1 and 2 is clockwise from three.

Exercise 3.1.2. Draw a different triangulation associated to the quiver in this example.

Okay great. We now have a way to get a triangulation of an $n+3$-gon for any quiver of type $\mathbb{A}_n$. Since the representations of these quivers are uniquely determined by their dimension vectors, Lemma 3.1.1, we can also represent all the indecomposable representations as diagonals on this triangulation! Let M_γ be the indecomposable representation of a quiver Q of type $\mathbb{A}_n$ whose dimension vector is given by $(0,0,\ldots,0,1,1,\ldots,1,0,0\ldots,0)$ where the 1's are located in spots $i, i+1, \ldots, i+k$. Then this representation is represented on the polygon by a diagonal γ that crosses *only* the diagonals labeled $i, i+1, \ldots, i+k$.

Lemma 3.1.2. *Fix a triangulation T_Q associated to the quiver Q of type $\mathbb{A}_n$. The association described above of $\gamma \leftrightarrow M_\gamma$ is a bijection.*

Example 3.1.4. Continuing with the quiver from the previous example, consider the representation $P(1) = {}^{1}_{2}$, whose dimension vector is $(1, 1, 0)$. Then the diagonal associated to this representation is the one that crosses both the diagonals labeled 1 and 2:

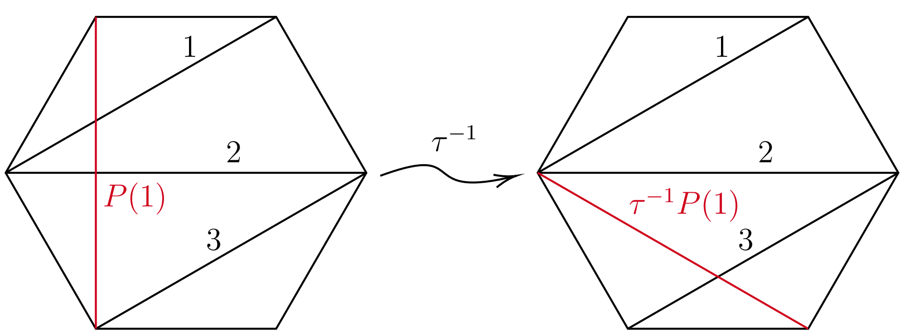


Now this next theorem encapsulates what I find so beautiful about mathematics. Recall how we find the AR translate. We needed 63 pages of mathematics just to define it! We need to take a representation, find a minimal projective resolution, apply the Nakayama functor, and find the kernel of a morphism in the abelian $\Bbbk$-category rep$Q$. This is a lot of deep ideas and a complicated mathematical notion. This next theorem captures all of this deep mathematics into the simple idea of rotating a polygon... Wow!

Theorem 3.1.3. *Let Q be a quiver of type $\mathbb{A}$, M_γ an indecomposable Q representation, T_Q, the triangulation associated to Q, and γ the unique diagonal on T_Q representing M_γ. Then the AR translate τM_γ is the representation corresponding to an elementary clockwise rotation of T_Q.*

What we mean by an elementary clockwise rotation is that we remove the labeled diagonals, but keep the diagonal γ. If the polygon has $n + 3$ vertices, we do a $\frac{2\pi}{n+3}$ clockwise rotation of the polygon. Then we put the labeled diagonals back into the diagram *the same way they were before the rotation*. The arc γ will now cross new diagonals and corresponds to τ.

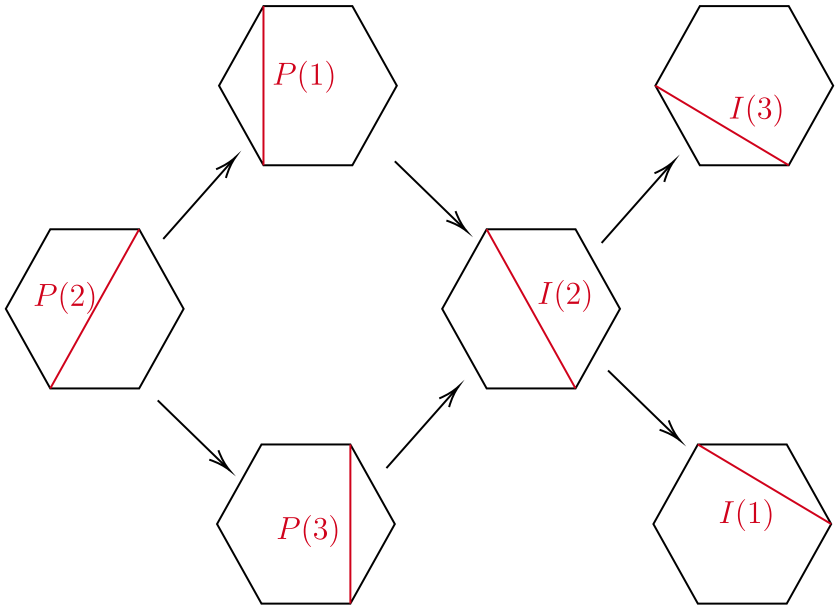
Example 3.1.5. Let's continue the running example. We can find $\tau^{-1}P(1)$ by rotating the above polygon counter clockwise as follows:



From this we see that $\tau^{-1}P(1) = S(3)$.

But wait! If we know how to realize τ and τ^{-1} as rotations of a polygon, we can compute the τ-orbits using these polygons and therefore construct the entire AR quiver!

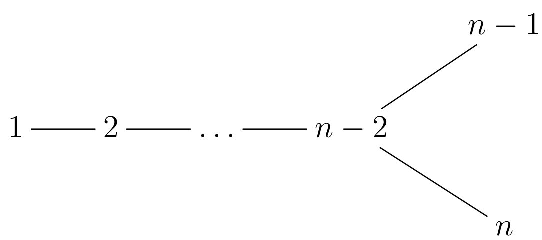
Example 3.1.6. Below is the AR quiver for $Q = 1 \to 2 \leftarrow 3$ in terms of polygons. We delete the labeled diagonals for simplicity. This AR quiver should look familiar, we almost computed it in Example 2.4.3!



3.2 Quivers of Type $\mathbb{D}$

Question 3.2. How do we compute AR quivers for type $\mathbb{D}$ quivers?

Definition 3.2.1. A **quiver of type** $\mathbb{D}_n$ is one whose underlying graph is



We will spend this section discussing how to compute the AR quiver for quivers of type $\mathbb{D}_n$. Just as with type $\mathbb{A}$ quivers, we will explain three methods.

Method 3.2.1. τ**-orbits**

Again, the projectives must be all the way on the left in the AR quiver and for each projective we repeatedly compute its inverse AR translate until it turns into an injective representation. The main difference in the AR quivers between type $\mathbb{A}$ and type $\mathbb{D}$ quivers is the existence of a new mesh, the type 4 mesh in Figure 1. This mesh causes there to be more than one projective on a horizontal strip, contrary to the case in type $\mathbb{A}$. Let's see an example.

Example 3.2.1. Consider the quiver Q given by

$$\begin{tikzcd}
 & & 3 \\
1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\
 & & 4
\end{tikzcd}$$

The projectives are $P(1) = \begin{smallmatrix}1\\2\\34\end{smallmatrix}, P(2) = \begin{smallmatrix}2\\34\end{smallmatrix}, P(3) = 3$, and $P(4) = 4$. Since there is an arrow from 1 to 2, 2 to 3, and 2 to 4, we order the projective representations as follows:

$$\begin{tikzcd}
3 \arrow[dr] & & \\
 & \begin{smallmatrix}2\\34\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\2\\34\end{smallmatrix} \\
4 \arrow[ur] & &
\end{tikzcd}$$

We see now that there are two projectives on the same horizontal strip in the AR quiver, which is different from the case of type $\mathbb{A}$. Let's compute $\tau^{-1}P(2)$. To do this, we have a minimal injective resolution

$$0 \to \begin{matrix}2\\34\end{matrix} \to I(3) \oplus I(4) \to I(1) \oplus I(2) \to 0.$$

Applying the Nakayama functor, we get an exact sequence

$$P(3) \oplus P(4) \to P(1) \oplus P(2) \to \begin{smallmatrix}1\\22\\34\end{smallmatrix} \to 0$$

where $\begin{smallmatrix}1\\22\\34\end{smallmatrix}$ is the cokernel of the map $3 \oplus 4 \to \begin{smallmatrix}1\\2\\34\end{smallmatrix} \oplus \begin{smallmatrix}2\\34\end{smallmatrix}$. Our updated AR quiver is:

$$\begin{tikzcd}
3 \arrow[dr] & & & \\
 & \begin{smallmatrix}2\\34\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\2\\34\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\22\\34\end{smallmatrix} \\
4 \arrow[ur] & & &
\end{tikzcd}$$

But wait! This means that $\begin{smallmatrix}2\\34\end{smallmatrix} \to \begin{smallmatrix}1\\2\\34\end{smallmatrix} \to \begin{smallmatrix}1\\22\\34\end{smallmatrix}$ by definition is part of a new mesh! Continuing right, we will see that for quivers of type $\mathbb{D}$, the representations in a mesh can all lie on the same horizontal strip; yet another difference from type $\mathbb{A}$. Continuing with this τ orbit, we compute $\tau^{-1} \begin{smallmatrix}1\\2\\34\end{smallmatrix} = 2, \tau^{-1}2 = \begin{smallmatrix}1\\2\end{smallmatrix}$, and $\tau^{-1}\begin{smallmatrix}1\\2\end{smallmatrix} = 1$, which is an injective so we're done. The updated AR quiver is then

$$\begin{tikzcd}
3 \arrow[dr] & & & & & & \\
& \begin{smallmatrix}2\\34\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\2\\34\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\22\\34\end{smallmatrix} \arrow[r] & 2 \arrow[r] & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[r] & 1 \\
4 \arrow[ur] & & & & & &
\end{tikzcd}$$

Computing the remaining two τ-orbits and inserting the remaining arrows gives us the full AR-quiver.

$$\begin{tikzcd}
3 \arrow[dr] & & \begin{smallmatrix}2\\4\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[dr] & & \\
& \begin{smallmatrix}2\\34\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}1\\2\\34\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\22\\34\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & 2 \arrow[r] & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[r] & 1 \\
4 \arrow[ur] & & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}1\\2\\4\end{smallmatrix} \arrow[ur] & &
\end{tikzcd}$$

Notice that the representations $\begin{smallmatrix}1\\2\\34\end{smallmatrix} \to \begin{smallmatrix}1\\22\\34\end{smallmatrix} \to 2$ form a mesh where all the representations lie on the same horizontal strip!

Wait! Why aren't there any arrows pointing straight up or down?

Method 3.2.2. The Knitting Algorithm

We will now present the knitting algorithm for type $\mathbb{D}$, which is quite similar to that of type $\mathbb{A}$. One key difference is that there are technically two new meshes we did not see in type $\mathbb{A}$ as seen in Figure 3.

(iv)
$$\begin{tikzcd}
& E_1 & & & E_1 \arrow[dr] & \\
\tau M \arrow[ur] \arrow[r] \arrow[dr] & E_2 & \rightsquigarrow & \tau M \arrow[ur] \arrow[r] \arrow[dr] & E_2 \arrow[r] & M \\
& E_3 & & & E_3 \arrow[ur] &
\end{tikzcd}$$

(v)
$$\begin{tikzcd}
M \arrow[r] & E_1 & \rightsquigarrow & M \arrow[r] & E_1 \arrow[r] & \tau^{-1}M
\end{tikzcd}$$

Figure 3: The Two *New* Types of Knitting in Type $\mathbb{D}$

Just like for type $\mathbb{A}$, the knitting algorithm for type $\mathbb{D}$ relies on the fact that each representation of a type $\mathbb{D}$ quiver is uniquely determined by it's dimension vector. though the association is not as straight forward as it is for type $\mathbb{A}$.

Lemma 3.2.1. *Any indecomposable representation of a type $\mathbb{D}_n$ quiver is uniquely determined by its dimension vector (up to isomorphism).*

Sketch of Proof. Let $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ be a dimension vector of an indecomposable representation. Then $d_i = 0, 1,$ or 2 for all i. Moreover, if we have that $d_i = 2$, there can only be certain values for i. Suppose that i is minimal, so d_i is the *first* occurrence of a 2 in $\boldsymbol{d}$. Then we have:

1. $i = 2, 3, \ldots,$ or $n - 2$, so the first entry can't be a 2.
2. For all vertices j such that $i \leq j \leq n-2$, $d_j = 2$. So there is a string of 2's all the way until the vertex $n-2$.
3. $d_{i-1} = 1 = d_{n-1} = d_n$. So the entry before the first 2 must be a 1 and the last two entries must also be 1.

In terms of the graph of type $\mathbb{D}_n$, representations look like this, where the number at each vertex is the dimension of the vector space at that vertex:

$$\begin{tikzcd}
 & & & & & & & & & 1 \\
0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \arrow[r, dash] & 1 \arrow[r, dash] & \cdots \arrow[r, dash] & 1 \arrow[r, dash, "\alpha_i"] & 2 \arrow[r, dash] & \cdots \arrow[r, dash] & 2 \arrow[ur, dash, "\beta_1"] \arrow[dr, dash, "\beta_2"'] & \\
 & & & & & & & & & 1
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & & & & & & 1 \\
0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \arrow[r, dash] & 1 \arrow[r, dash] & \cdots \arrow[r, dash] & 1 \arrow[r, dash] & 1 \arrow[r, dash] & \cdots \arrow[r, dash] & 1 \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & & & & & & & 1
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & & & & & & 0 \\
1 \arrow[r, dash] & \cdots \arrow[r, dash] & 1 \arrow[r, dash] & 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & & & & & & & 0
\end{tikzcd}$$

We define the corresponding representation $M_{\boldsymbol{d}} = (M_i, \varphi_\alpha)$ to have vector spaces $M_i = \mathbb{k}^{d_i}$. Now if either (or both) of the vector spaces $M_{s(\alpha)}$ or $M_{t(\alpha)}$ are zero, φ_α is the zero map. If both $M_{s(\alpha)}$ or $M_{t(\alpha)}$ are nontrivial and of the same dimension, then φ_α is the identity map. There is however one other possible option, namely when one vector space has dimension one and the other has dimension 2. There are precisely three such arrows, $\alpha_i, \beta_1,$ and β_2 depicted above. Since all the maps from vertex $i+1$ to vertex $n-2$ are the identity, in order to make sure the representation is indecomposable, we need to keep track of the one dimensional subspace associated to the map φ_{α_i}. So consider the one dimensional subspace of M_{i+1} given by

$$\begin{cases} \mathrm{im}\varphi_{\alpha_i} & \text{if } \alpha_i \text{ points to } i+1 \\ \mathrm{ker}\varphi_{\alpha_i} & \text{otherwise} \end{cases}$$

Then under the composition of the maps (or their inverses) $\varphi_{\alpha_{n-3}} \circ \cdots \circ \varphi_{\alpha_{i+1}}$, this subspace is sent to a one dimensional subspace of M_{n-2}. Let's call this l_1. We also have two other one-dimensional subspaces of M_{n-2}, namely,

$$l_2 = \begin{cases} \mathrm{im}\varphi_{\beta_1} & \text{if } \beta_1 \text{ points to } n-2 \\ \mathrm{ker}\varphi_{\beta_1} & \text{otherwise} \end{cases} \quad l_3 = \begin{cases} \mathrm{im}\varphi_{\beta_2} & \text{if } \beta_2 \text{ points to } n-2 \\ \mathrm{ker}\varphi_{\beta_2} & \text{otherwise} \end{cases}$$

The condition that we impose on the three maps $\varphi_{\alpha_i}, \varphi_{\beta_1}$, and φ_{β_2} is that the subspaces l_1, l_2, and l_3 are pairwise distinct. This gives, up to isomorphism, an indecomposable representation with dimension vector $\boldsymbol{d}$. □

Okay great, we have that the dimension vectors uniquely define the representations. If the above construction of the representation is confusing, that's okay. We won't be looking very closely at the linear transformations that make up the representation for the remainder of this course. But now, we can knit the AR-quiver *exactly* the way we did in the knitting algorithm for type $\mathbb{A}$ quivers, Definition 3.1.3. The only difference is that we have new meshes to keep track of.

Example 3.2.2. Let Q be the $\mathbb{D}_5$ quiver:

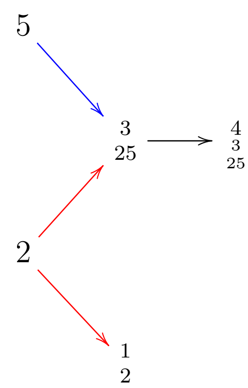
$$\begin{tikzcd} & & & 4 \arrow[dl] \\ 1 \arrow[r] & 2 & 3 \arrow[l] \arrow[dr] & \\ & & & 5 \end{tikzcd}$$

Step 1 Compute the indecomposable projectives:

We have the projectives: $P(1) = \begin{smallmatrix}1\\2\end{smallmatrix}, P(2) = 2, P(3) = \begin{smallmatrix}3\\25\end{smallmatrix}, P(4) = \begin{smallmatrix}4\\3\\25\end{smallmatrix}$, and $P(5) = 5$.

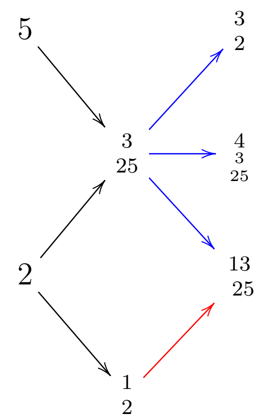
Step 2 Line up the projectives in their partial order:

We have an arrow from $P(2)$ into both $P(1)$ and $P(3)$. We also have an arrow from $P(3)$ into $P(4)$ and an arrow from $P(5)$ to $P(3)$. Since there are four arrows in the quiver, we have found all the arrows between projectives in the AR quiver. Here is one way to start the AR quiver:

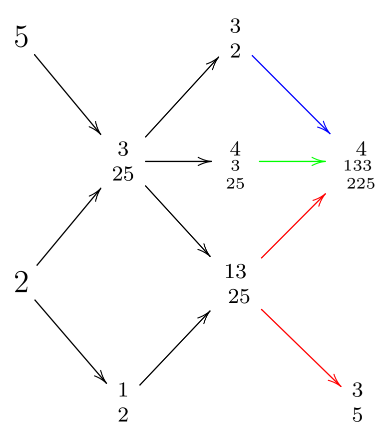


Step 3 Knit the meshes:

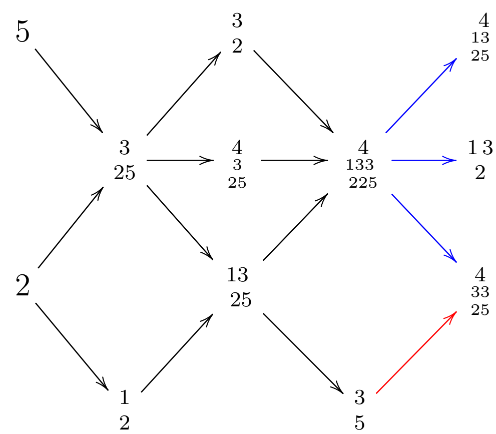
There are two meshes that can be knitted here: the blue one and the red one. We knit them the exact same way as we did for type $\mathbb{A}$ to get:



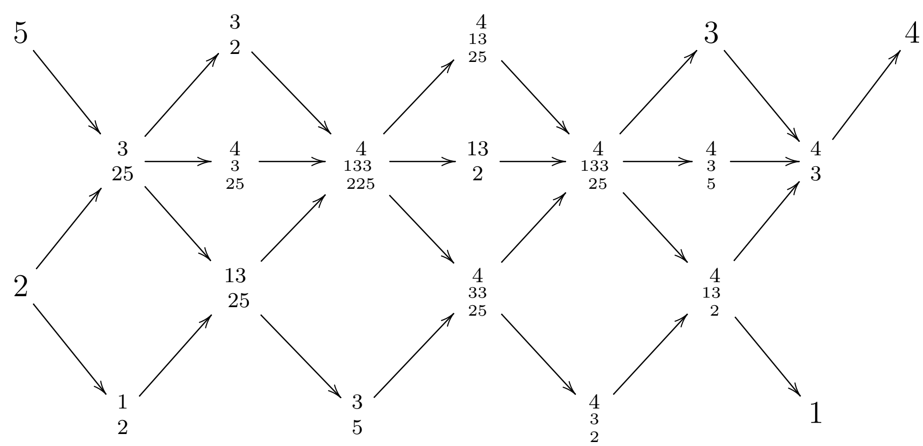
Step 4 Repeat step 3 to get:



Step 4 Repeat step 3. Notice here there are *three* meshes to complete: the blue, green, and red!



Step 4 Repeat step 3 until we finish with the AR quiver:



Method 3.2.3. Triangulations of a Punctured n-gon

In this section, we will introduce another geometric interpretation of the AR quiver, similar to the one introduced for type $\mathbb{A}$. Since the indecomposable representations of type $\mathbb{D}$ are uniquely determined by their dimension vectors, we can use arcs on some surface to represent them. As the reader may guess, since the association of dimension vector to representation is more complicated for type $\mathbb{D}$ than for type $\mathbb{A}$, so will the geometric interpretation. To each quiver of type $\mathbb{D}_n$, we will be studying a **punctured n-gon**. This is precisely a regular n-gon with a dot in the middle representing a puncture.

On this punctured n-gon, we can draw at least three types of **arcs**:

- Arcs that connect two vertices of the punctured n-gon. These are called **exterior arcs**.
- Arcs that connect the puncture to a vertex. These are called **plain bridging arcs**.
- Tagged arcs that connect the puncture to a vertex. These are called **tagged bridging arcs**.

An arc (tagged or plain) is called a **bridging arc** if one of its endpoints is the puncture. Tagged arcs are straight lines from the puncture to a vertex decorated with a small little notch. Remember that we will associate each of these arcs to an indecomposable representation of a quiver of type $\mathbb{D}$. The reason we need notched arcs probably isn't clear right now but we will see why soon. An example of what we're talking about is in Figure 4.

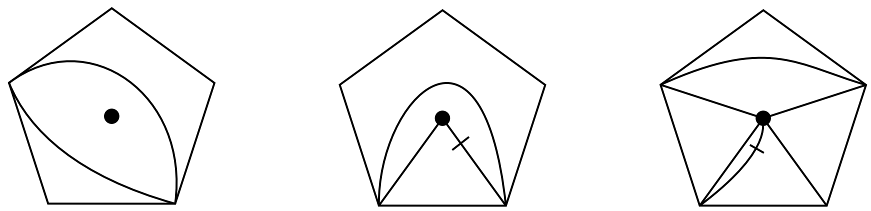


Figure 4: Examples of Arcs on a Punctured 5-gon

Recall that a **triangulation** is a *maximal* collection of non-crossing arcs. In type $\mathbb{A}$, arcs (diagonals) crossed when they crossed! That is, two diagonals crossed if and only if they crossed when drawn in the polygon. This seems like the intuitive definition, but in the case of punctured polygons, we define crossing arcs differently. Think of it like changing the set of rules by which we play our game. What new things can we come up with if we play by these new rules?

To define crossing arcs formally, we need the notion of isotopy or homotopy, which is beyond the scope of this course. The upshot here though is that for arcs on the punctured polygon γ and γ', if one of the arcs is an exterior arc, then the two arcs cross if and only if they cross when we draw them. Let the **crossing number**, denoted by $e(\gamma, \gamma')$, be the number of times the arcs γ and γ' cross. The less intuitive case is when both γ and γ' are bridging arcs. Suppose that a and a' are the endpoints of γ and γ' on the boundary of the punctured n-gon respectively. In this case we define the crossing number as follows:

$$e(\gamma, \gamma') = \begin{cases} 0 & \text{if } \gamma \text{ and } \gamma' \text{ are both plain bridging arcs} \\ 0 & \text{if } \gamma \text{ and } \gamma' \text{ are both tagged bridging arcs} \\ 0 & \text{if } a = a' \\ 1 & \text{if } \gamma \text{ and } \gamma' \text{ have opposite tagging and } a \neq a' \end{cases}$$

An example of crossing numbers can be seen in Figure 5.

Okay, now that we know the rules to the game, we can construct some maximal collections of non-crossing arcs. In type $\mathbb{A}$, triangulations always split the polygon into triangles. In type $\mathbb{D}$ however, this need not be the case as can be seen in Figure 6.

Just like for type $\mathbb{A}$, we want to associate a triangulation of a punctured n-gon to a quiver of type $\mathbb{D}$. Then we will see how many times arcs cross this triangulation, which will give us the dimension vector of an indecomposable representation. Let Q be a quiver of type $\mathbb{D}_n$. We begin the triangulation T_Q with an exterior arc γ_1 that cuts off a triangle. We can choose where this arc goes, so this association is certainly not unique. Then, just like for type $\mathbb{A}$, if there is an arrow $2 \to 1$ in Q, then we draw the *unique* exterior arc γ_2 that is counter

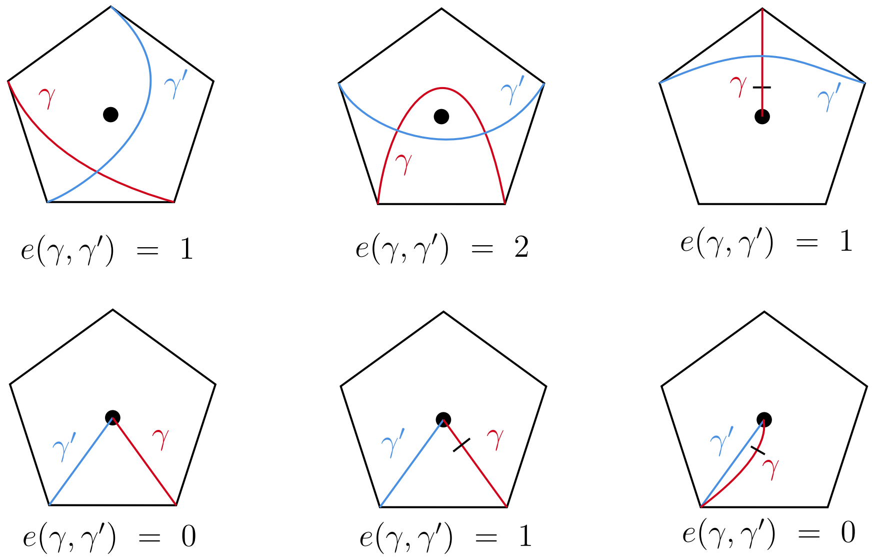


Figure 5: Some Crossing Numbers

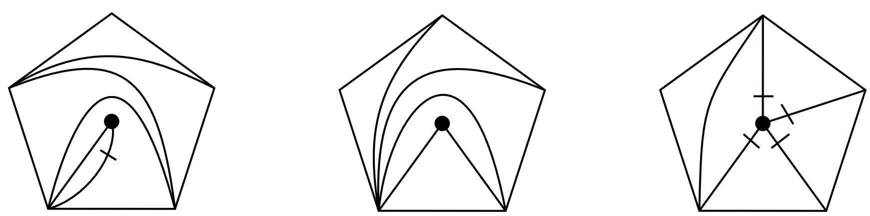


Figure 6: Some Triangulations

clockwise from γ_1 and cuts off a triangle. If there is an arrow $1 \to 2$ in Q, then we draw the *unique* exterior arc γ_2 that is clockwise from γ_1 and cuts off a triangle. We continue this process until we get to the vertex $n-2$. It is here where the quiver is fundamentally different from a type $\mathbb{A}$ quiver, and hence the triangulation process will also be different. There are 4 possibilities for what can happen at this vertex. All four possibilities along with the corresponding triangulations is shown below in Figure 7.
Let's see an example of a triangulation associated to a quiver.

Example 3.2.3. Let Q be the quiver

$$\begin{tikzcd} & & & 4 \arrow[dl] \\ 1 \arrow[r] & 2 & 3 \arrow[l] \arrow[dr] & \\ & & & 5 \end{tikzcd}$$

One triangulation associated to this quiver is the following:

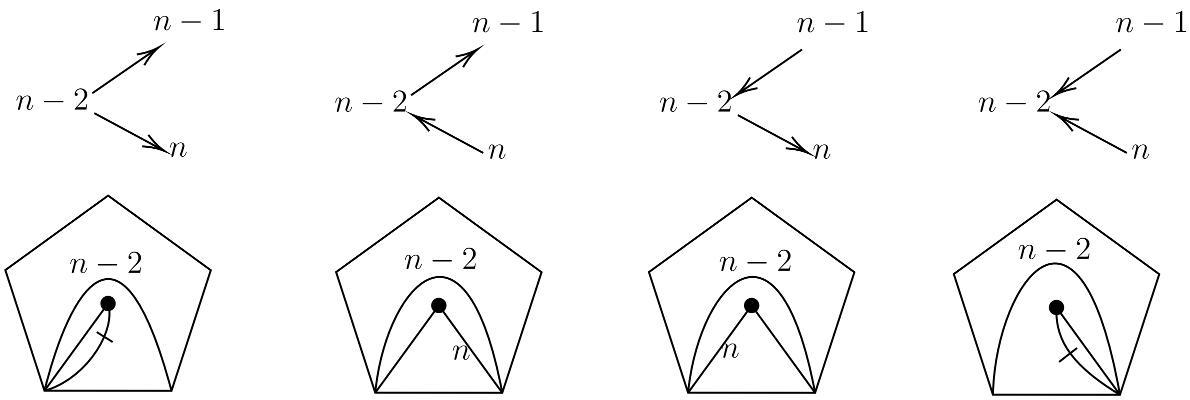
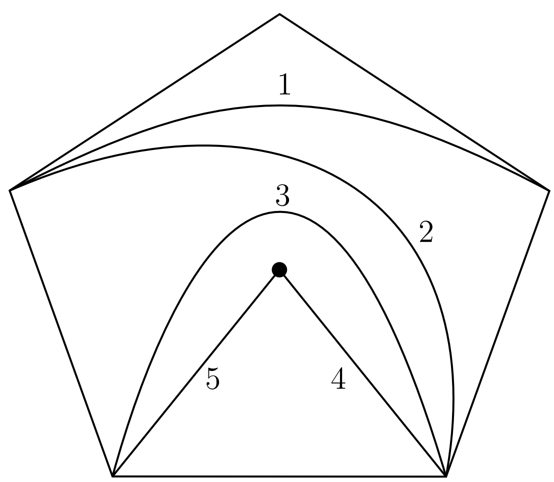


Figure 7: The Four Possibilities to Finish The Triangulation T_Q

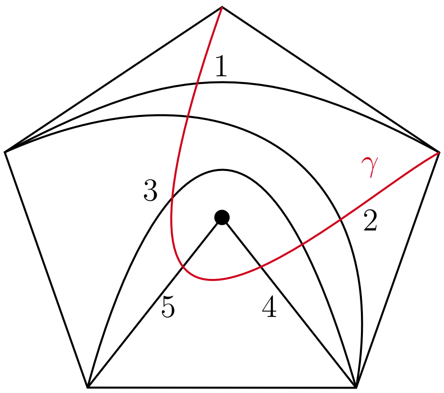


Now given a triangulation T_Q, any arc that is not already in the triangulation will cross the arcs that are in the triangulation. Actually, any such arc is uniquely defined by the set of diagonals that the arc crosses. Suppose γ is an arc in a punctured n-gon and fix a triangulation T_Q of this n-gon where the arcs in the triangulation are labeled 1 through n. Define $\boldsymbol{d_\gamma} = (e(\gamma, 1), e(\gamma, 2), \ldots, e(\gamma, n))$. Then we have the following lemma which we will not prove:

Lemma 3.2.2. *Let M_γ be the unique representation with dimension vector $\boldsymbol{d_\gamma}$. Then there is a bijection between arcs on the punctured n-gon with triangulation T_Q and indecomposable representations of Q given by $\gamma \mapsto M_\gamma$.*

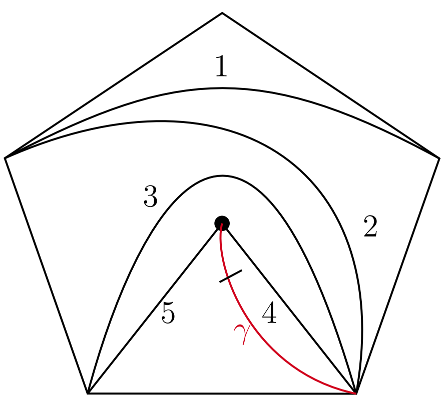
Let's see an example.

Example 3.2.4. Let's continue with the quiver and triangulation from Example 3.2.3. Then we have the following arcs and corresponding representations:



$$\longrightarrow d_\gamma = (1,2,2,1,1) \longrightarrow M_\gamma = \begin{smallmatrix} 4 \\ 133 \\ 225 \end{smallmatrix}$$

Now looking at this triangulation, there is *no way* to draw an exterior arc that only crosses arc 5 in the triangulation because it will also cross 3. But there is an indecomposable representation whose dimension is given by $(0,0,0,0,1)$, it is the simple at 5! Aha! *This* is why we need these notched arcs! Representing the simple at 5, we have the following:



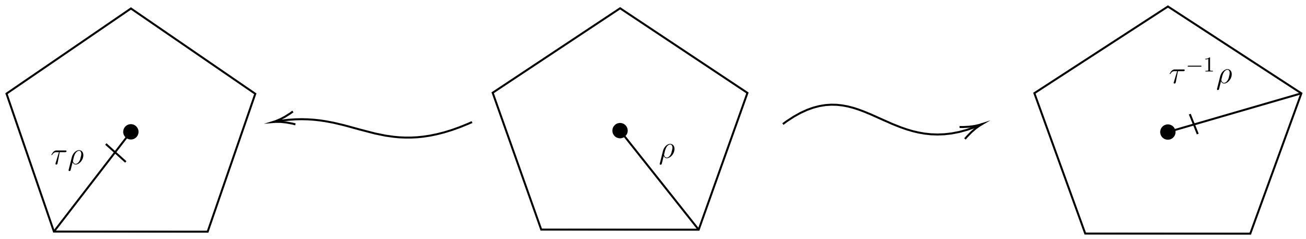
$$\longrightarrow d_\gamma = (0,0,0,0,1) \longrightarrow M_\gamma = 5$$

Now why is this so cool? Well because we have an analogous theorem to that of Theorem 3.1.3!

Theorem 3.2.3. *Let Q be a quiver of type $\mathbb{D}_n$ and T_Q a triangulation of a punctured n-gon. The AR translate τ is given by elementary clockwise rotation of the punctured polygon with simultaneous change of tags.*

Using this theorem, we can give a geometric construction of the AR quiver by computing τ-orbits just like we did for type $\mathbb{A}$ quivers.

Example 3.2.5. Let's keep going with the quiver and triangulation from Example 3.2.3. Then for instance, we have the following:



By computing τ-orbits, we get the AR quiver: