Mini course: Curvature measures of singular spaces in the riemannian and pseudo-riemannian setting

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Abstract. The general aim of this mini-course is to show how the modern theory of valuations can be used in order to extend classical curvature notions (in particular scalar curvature) from (pseudo-)riemannian manifolds to certain singular, but tame sets.

The course consists of three lectures. In the first lecture, we introduce several classes of singular subsets of euclidean space or of manifolds: convex sets, sets of positive reach, manifolds with corners, differentiable polyhedra, semialgebraic and subanalytic sets, sets definable in some o-minimal structure. The main point is that such sets, although being singular, are tame enough to admit an Euler characteristic. We also introduce the normal cycle of such sets.

In the second lecture, we introduce intrinsic volumes in the euclidean/riemannian setting. The tube formulas by Steiner and Weyl will be discussed, as well as the modern valuation-theoretic approach to Weyl's principle. A special emphasis will be given to the total scalar curvature (Einstein-Hilbert functional), which is one of the intrinsic volumes. It can be linked to curvature bounds in the sense of metric geometry (Alexandrov spaces). Moreover, the classical first variation formula for the total scalar curvature can be extended to the singular setting and yields a distributional Einstein tensor of singular spaces.

In the third lecture we explore how much of this theory carries over to the pseudo-euclidean and pseudo-riemannian case. In these cases, one needs generalized valuations, and I will spend some time explaining this notion. It turns out that one can still define intrinsic volumes, which are complex-valued generalized valuations. They satisfy a pseudo-riemannian version of the Weyl principle. This lecture will be followed by D. Faifman's talk, where an extension to generic submanifolds of pseudo-riemannian manifolds is discussed.

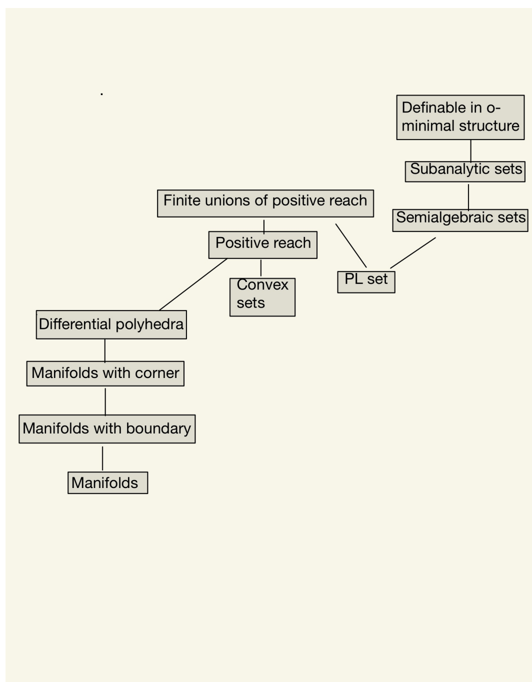
Contents

CHAPTER 1

Lecture I: Tame sets and smooth valuations

1. Families of tame sets

In the first section we introduce different classes of sets. The sets may be singular, but "tame" enough to admit for instance a Euler characteristic. Examples of sets that are not tame are fractals. For each class of tame sets, a notion of normal cycle is available that makes it possible to define certain curvature measures of such sets. This will be the content of the second lecture.



1.1. Manifolds with boundary or corners, differentiable polyhedra. A manifold is locally modelled on $\mathbb{R}^n$: the charts are maps $\phi : U \to \phi(U)$, where $\phi(U) \subset \mathbb{R}^n$ is an open set. Moreover, the coordinate change is smooth. A manifold with boundary is locally modelled over the half space $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geqslant 0\}$. The boundary of M is then the set of points that get mapped to the boundary of $\mathbb{R}^n_+$, and one can show that this property is independent of the coordinate chart. A manifold with corners is locally modelled over the octant $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1, \ldots, x_n \geqslant 0\}$. It admits a stratification by types, where the type of a point is determined by the number of zero coordinates. Finally, a differentiable polyhedron is locally modelled over a polyhedron in $\mathbb{R}^n$. It also admits a stratification by types, where the type is determined by the dimension of the face of the polyhedron.

1.2. Convex sets and sets with positive reach. Let V be a finite-dimensional real vector space. By $\mathcal{K}(V)$ we denote the set of all compact and convex bodies in V. Given a euclidean scalar product on V with unit ball B, we can define the Hausdorff distance between $K, L \in \mathcal{K}(V)$ by

$$d(K, L) = \inf\{\epsilon > 0 : K \subset L + \epsilon B, L \subset K + \epsilon B\}.$$

The corresponding topology on $\mathcal{K}(V)$ is independent of the choice of the scalar product. The space $\mathcal{K}(V)$ is closed under Minkowski sum

$$K + L = \{x + y : x \in K, y \in L\},$$

under intersection, and under projections. It is not closed under union. We refer to [**24**] for a detailed study of compact convex sets.

The set consisting of finite unions of compact convex bodies is closed under finite unions and intersections. Note however that the Hausdorff topology is not well suited in this case: take the boundary of a small square, which is the union of 4 segments, hence a finite union of compact convex bodies. As the side of the square tends to zero, the set converges to a point. However, the Euler characteristic of the boundary of a square is that of a circle, i.e. 0, while the Euler characteristic of a point is 1.

Federer has introduced the sets of positive reach.

Definition 1.1. *Let V be a euclidean vector space of finite dimension. A set $P \subset V$ is called set of positive reach if there is some $r > 0$ such that each point $x \in V$ whose distance to P is less than r has a unique foot point in P. The supremum over all such r is called the reach of P.*

Although the reach of P depends on the choice of the euclidean scalar product on V, the property of being of positive reach does not. More generally, one can define this notion on a riemannian manifold (or even in any metric space). A theorem by Bangert says that the image of a set of positive reach under a diffeomorphism is again of positive reach, in particular the notion "set of positive reach" is independent of the riemannian metric.

Every compact convex body is a set of positive reach, its reach is actually $+\infty$. Every compact submanifold (possibly with boundary or corners) and each differentiable polyhedron is of positive reach. The union X of two lines is not of positive reach.

To a set of positive reach, one can associate principal curvatures that depend on a point of the set and a normal direction of that point. These curvatures can take the value $+\infty$ (e.g. in the case of a corner), but not $-\infty$ [**31**]. Sets of positive reach are not closed under projections: take two lines in $\mathbb{R}^3$ that do not intersect, such that their projections in $\mathbb{R}^2$ intersect.

Finally, one can consider also finite unions of sets of positive reach [**22, 30, 23**].

1.3. Semialgebraic sets, subanalytic sets, o-minimal structures. We refer to [**9**] for semialgebraic sets and real algebraic geometry.

DEFINITION 1.2. *A semialgebraic subset of* $\mathbb{R}^n$ *is a finite union of sets of the form*

$$\{x \in \mathbb{R}^n | f_1(x) = \ldots = f_l(x) = 0, g_1(x) > 0, \ldots, g_m(x) > 0\}$$

where f_i, g_j *are polynomials.*

Example: the semialgebraic subsets of $\mathbb{R}$ are precisely the finite union of intervals.

By definition, finite unions of semialgebraic sets are semialgebraic and it is easy to see that finite intersections are semialgebraic again. The complement of a semialgebraic set is again semialgebraic: it is enough to check this for a set of the form

$$\{x \in \mathbb{R}^n | f_1(x) = \ldots = f_l(x) = 0, g_1(x) > 0, \ldots, g_m(x) > 0\},$$

whose complement is

$$\bigcup_{i=1}^{l} \{f_i(x) > 0\} \cup \{-f_i(x) > 0\} \cup \bigcup_{j=1}^{m} \{-g_j(x) > 0\} \cup \{g_j(x) = 0\}.$$

Semialgebraic sets are in general not of positive reach, and convex sets or sets of positive reach are in general not semialgebraic (however, there are important classes of sets that are convex and semialgebraic, like spectahedra).

THEOREM 1.3 (Tarski-Seidenberg). *Let* $\pi : \mathbb{R}^n \to \mathbb{R}^k$ *be the projection onto the first* k *coordinates. If* $X \subset \mathbb{R}^n$ *is semialgebraic, then* $\pi(X) \subset \mathbb{R}^k$ *is semialgebraic.*

The theorem can be stated in model theoretic terms: semialgebraic sets are those sets that are definable by a first order formula involving quantifiers (over $\mathbb{R}$), and polynomial equalities and inequalities. For instance, the closure of a semialgebraic set is semialgebraic again, as it can be expressed as

$$\bar{S} = \{y \in \mathbb{R}^n : \forall \epsilon > 0 \exists x \in S : |y - x| < \epsilon\}.$$

DEFINITION 1.4. *A semianalytic subset of* $\mathbb{R}^n$ *is a finite union of sets of the form*

$$\{x \in \mathbb{R}^n | f_1(x) = \ldots = f_l(x) = 0, g_1(x) > 0, \ldots, g_m(x) > 0\}$$

where f_i, g_j *are real analytic functions (meaning that in a neighborhood of each point, they agree with their Taylor expansion).*

This class is closed under finite unions, finite intersections, and complement. However, it is not closed under projection.

Definition 1.5. *A set $X \subset \mathbb{R}^n$ is subanalytic if there is some $N > n$ and a semianalytic set $Y \subset \mathbb{R}^N$ such that $X = \pi(Y)$.*

It is obvious that a semialgebraic set is semianalytic, and a semianalytic set is subanalytic. See [**8**] for more information on semianalytic and subanalytic sets.

A generalization is given by o-minimal structures. We refer to [**26**] or [**27**].

Definition 1.6. *An o-minimal structure is a sequence $S_0, S_1, \dots$ such that*

(1) *S_n is a Boolean algebra of subsets of $\mathbb{R}^n$ (i.e. closed under finite unions and intersections, and under complement).*
(2) *If $A \in S_n, B \in S_m$, then $A \times B \in S_{n+m}$.*
(3) *Real algebraic sets in $\mathbb{R}^n$ belong to S_n.*
(4) *If $X \in S_{n+1}$, then $\pi(X) \in S_n$, where π is the projection onto the first n coordinates.*
(5) *S_1 contains precisely the finite unions of intervals.*

A set belonging to some S_n is called definable.

Examples

(1) The semialgebraic sets define an o-minimal structure. Every other o-minimal structure contains this one.
(2) The subanalytic sets do not form an o-minimal structure ($\{x \in \mathbb{R} : \sin(x) = 0\}$ is not a finite union of intervals). However, globally subanalytic sets (i.e. sets that are subanalytic in $\mathbb{RP}^n$) form an o-minimal system. Alternatively, one can only allow restricted analytic functions (i.e. functions on $[0,1]^n$ that are analytic in a neighborhood of the cube) in the definition of semianalytic sets. This gives the same o-minimal structure.
(3) The sets that can be defined using polynomials, the exponential function, and quantifiers over the reals, such as
$$\{(x,y) \in \mathbb{R}^2 : \exists z \in \mathbb{R} \quad e^{e^z - x} < x + yz\}$$
are an o-minimal system.

We will need two ways of decomposing a definable set into easier pieces.

Theorem 1.7. *Let X be definable in some o-minimal structure. Then*

(1) *X admits a Nash stratification, that is $X = \bigcup S_\alpha$, where each S_α is a definable connected subvariety such that if $S_\alpha \cap \bar{S}_\beta \neq \emptyset$ for $\alpha \neq \beta$, then $S_\alpha \subset \bar{S}_\beta$ and $\dim S_\alpha < \dim S_\beta$. We let $\mathcal{S}_d$ be the union of all strata of dimension d.*
(2) *X can be written as a finite disjoint union of definable sets that are (definably) homeomorphic to some open cube $(0,1)^d$.*

It follows that one can define the Euler characteristic of a compact definable set X by $\chi(X) = \sum_d (-1)^d n_d$, where n_d is the number of cubes of dimension d. In the non-compact case, one can define two different versions of the Euler characteristic (corresponding to cohomology or to cohomology with compact support). We will restrict to the compact case.

COROLLARY 1.8. *The Euler characteristic is finitely additive in the sense that if X, Y are compact definable sets, then*

$$\chi(X \cup Y) + \chi(X \cap Y) = \chi(X) + \chi(Y).$$

1.4. PL-sets. Let K be a simplicial complex. To each simplex σ in K, we associate a simplex $\tilde{\sigma}$ of the same dimension in some euclidean space. If $\sigma' \subset \sigma$, then the corresponding map $\tilde{\sigma}' \to \tilde{\sigma}$ should be an isometric embedding. The corresponding metric space X is called a PL-space (piecewise linear). Intrinsic volumes of such spaces were considered in [**14**].

2. Normal cycle construction

Let K be a compact convex body in a vector space V. A support element is a pair (x, H), where $x \in \partial K$ and H is a support plane of K at x. This means that H is a cooriented hyperplane such that K is contained in the negative closed affine half-space $x + H_-$. Let us assume that V is a euclidean vector space. If $y \in V \backslash K$, then there is a unique foot point $x \in \partial K$ closest to y. Then x together with the hyperplane $(y - x)^\perp$, oriented in such a way that $y \in x + H_+$, is a contact element. If we identify H with the unit vector orthogonal to H and pointing in the positive direction, we can think of a contact element as an outward pointing normal vector of ∂K. However, the notion of a contact element (x, H) does not use a euclidean scalar product.

Let $\mathrm{Gr}_{n-1}^+(TV)$ be the set of all pairs (x, H) with $x \in V$ and $H \subset V$ a cooriented hyperplane. Note that this is a smooth manifold, in fact the product of V and the Grassmann manifold of cooriented hyperplanes.

The set $\mathrm{Nor}\, K$ of all contact elements is an oriented Lipschitz submanifold of $\mathrm{Gr}_{n-1}^+(TV)$ of dimension $(n-1)$: consider the set K_r of all points $y \in V$ with distance to K equal to some fixed $r > 0$, and map y to the contact element that we have described above. If r is chosen at random (outside some set of measure zero of critical values, cf. Sard's theorem), then K_r is a Lipschitz manifold, so we have a Lipschitz parametrization of $\mathrm{Nor}(K)$.

We think of $\mathrm{Nor}(K)$ as an $(n-1)$-dimensional current, i.e. a functional on the space $\Omega^{n-1}(\mathrm{Gr}_{n-1}^+(TV))$ of differential $(n-1)$-forms on $\mathrm{Gr}_{n-1}^+(TV)$.

DEFINITION 2.1. $\mathrm{Nor}\, K$ *is called the normal cycle of* K.

It is easy to check that $\mathrm{Nor}(K)$ is indeed a cycle, i.e. vanishes on exact forms.

LEMMA 2.2. *If K, L are compact convex bodies such that $K \cup L$ is convex as well, then*

$$\mathrm{Nor}(K \cup L) + \mathrm{Nor}(K \cap L) = \mathrm{Nor}(K) + \mathrm{Nor}(L).$$

Definition 2.3. *A functional* $\mu : \mathcal{K}(V) \to \mathbb{R}$ *of the form*

$$\mu(K) = \int_K \phi + \int_{\operatorname{Nor} K} \omega, \quad \phi \in \Omega^n(V), \omega \in \Omega^{n-1}(\mathrm{Gr}_{n-1}^+(TV))$$

is called a smooth valuation.

Corollary 2.4. *A smooth valuation* μ *is finitely additive in the sense that*

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)$$

whenever $K, L, K \cup L$ *are compact convex sets.*

A functional μ that has this property is called a valuation. Valuations on polytopes were first used by Dehn in his solution of Hilbert's 3rd problem. They were highly non continuous (and hence not smooth). In integral geometry, the space Val of translation invariant, continuous valuations plays an important role. The subspace of smooth valuations is dense and carries a lot of algebraic structure (a product satisfying a version of Poincaré duality, a version of the Hard Lefschetz theorem, a convolution product, a Fourier transform relating product and convolution...).

Important fact 2.5. *For each class of tame sets described above, there is a normal cycle construction. In particular, a smooth valuation* μ *can be extended to such tame sets by setting*

$$\mu(X) = \int_X \phi + \int_{\operatorname{Nor} X} \omega.$$

In the case of sets of positive reach, the construction is as for convex bodies, except that r should be chosen smaller than the reach of X. In the case of semialgebraic sets or sets belonging to an o-minimal system, one has to use stratified Morse theory to attach certain integer multiplicities to contact elements in such a way that the resulting current is a cycle, see [**11**].

CHAPTER 2

Lecture II: Curvatures of tame sets

1. Steiner's formula

Let ω_i denote the volume of the i-dimensional unit ball.

THEOREM 1.1 (Steiner [**25**]). *Let V be a euclidean vector space of dimension n and $K \in \mathcal{K}(V)$. Then there are real numbers $\mu_0(K), \ldots, \mu_n(K)$ such that the volume of the r-tube around K is given by*

$$\mathrm{vol}(K + rB) = \sum_{k=0}^{n} \mu_k(K)\omega_{n-k}r^{n-k}.$$

The $\mu_i, i = 0, \ldots, n$ are continuous valuations, called intrinsic volumes. Moreover, if $V \subset W$ is an isometric embedding, then $\mu_k^W|_V = \mu_k^V$ (with the convention that $\mu_k^V = 0$ if $k > \dim V$).

Examples:

- $\mu_0(K) = 1$. For reasons that will become clear afterwards, μ_0 is called Euler characteristic.
- $\mu_n(K) = \mathrm{vol}(K)$.
- $\mu_{n-1}(K) = \frac{1}{2}\mathrm{vol}_{n-1}(\partial K)$.
- If the boundary of K is smooth, then

  $$\mu_k(K) = \frac{1}{(n-k)\omega_{n-k}} \int_{\partial K} \sigma_{n-1-k}(x)dx,$$

  where $\sigma_{n-1-k}(x)$ is the $(n-1-k)$th elementary symmetric function of the principal curvatures at x.

Let Val denote the vector space of continuous translation invariant valuations on $\mathcal{K}(V)$. This is an infinite-dimensional space, in fact a Banach space. Let $\mathrm{Val}^{\mathrm{SO}(n)}$ be the subspace of rotation invariant elements. Clearly $\mu_k \in \mathrm{Val}^{\mathrm{SO}(n)}$.

THEOREM 1.2 (Hadwiger [**19**]). *The vector space $\mathrm{Val}^{\mathrm{SO}(n)}$ is spanned by $\mu_0, \ldots, \mu_n$, in particular it is of dimension $(n+1)$.*

It is rather easy to show that the μ_k are smooth valuations in the sense of Definition 2.3 (just write down the translation- and rotation invariant differential forms on the sphere bundle). It follows that they can be evaluated on any tame set. There are also tube formulas in this setting, but one has to be careful about multiplicities. As an example, let $X \subset \mathbb{R}^n$ be a compact semialgebraic set. Then

$$\int_{\mathbb{R}^n} \chi(X \cap B(x,r))dx = \sum_{k=0}^{n} \mu_k(X)\omega_{n-k}r^{n-k}.$$

Note that the left hand side is just the volume of the r-tube if X is convex.

We will also need a local version of Steiner's formula. Let $U \subset V$ be a Borel subset. Inside the r-tube $K + rB$, consider only those points whose foot points on K belong to U. Then

$$\mathrm{vol}((K + rB) \cap \pi^{-1}(U)) = \sum_{k=0}^{n} \Phi_k(K,U)\omega_{n-k}r^{n-k},$$

and the coefficient $\Phi_k(K,U)$ is called Lipschitz-Killing curvature measure. For fixed K, the map $U \mapsto \Phi_k(K,U)$ is a measure.

2. Weyl's tube formula and enhanced Weyl's principle

THEOREM 2.1 (Weyl [**29**]). *Let $M \subset \mathbb{R}^N$ be a compact submanifold of dimension n, possibly with boundary. Then the volume of the r-tube around M is given, for small $r > 0$, by a polynomial*

$$\mathrm{vol}_N(M_r) = \sum_{k=0}^{n} \mu_k(M)\omega_{N-k}r^{N-k}.$$

The $\mu_k(M)$ do not depend on the embedding, but only on the inner geometry of the Riemanian manifold (M,g).

Examples: Suppose that $\partial M = \emptyset$.

- $\mu_0(M) = \chi(M)$.
- $\mu_n(M) = \mathrm{vol}_n(M)$.
- Important: $\mu_{n-2}(M) = \frac{1}{4\pi}\int_M s d\,\mathrm{vol}$, the total scalar curvature of M. Recall that it plays an important role in GR (Hilbert-Einstein functional).
- $\mu_k(M) = 0$ if $(n-k)$ is odd (if M has a boundary, there are boundary contributions, depending on the second fundamental form of the boundary).
- $\mu_k(M)$ can be written as the integral over some (rather complicated) polynomial in the Riemann curvature tensor of (M,g).
- $\mu_k(M)$ is a spectral invariant of the Laplacian acting on differential forms [**16**]).

Let $X \subset M$ be a tame set. For instance, X may be of positive reach, a manifold with corners, or - in case M is real analytic - a subanalytic subset. We use Nash's embedding theorem to find an isometric embedding $\iota : (M,g) \hookrightarrow \mathbb{R}^N$ for some N. ιX is then a tame subset of $\mathbb{R}^N$ and as such it has intrinsic volumes $\mu_k(\iota X)$. We set $\mu_k^M(X) := \mu_k(\iota X)$. One can then show that this is independent of the choice of ι. The functional $X \mapsto \mu_k^M(X)$

is a smooth valuation in the sense below. Similarly, given X and a Borel subset $U \subset M$, we can define $\Phi_k^M(X,U) := \Phi_k(\iota X, \iota U)$. This is a smooth curvature measure in the sense below.

Let $\mathrm{Gr}_{n-1}^+(TM)$ be the fiber bundle over M such that the fiber over a point $p \in M$ is the Grassmannian of cooriented hyperplanes in T_pM. In the case of a riemannian manifold, this can be identified with the sphere bundle over M.

The normal cycle construction from the previous talk extends to this situation and gives an $(n-1)$-dimensional cycle in $\mathrm{Gr}_{n-1}^+(TM)$.

Definition 2.2. *Let $\mathcal{P}(M)$ some class of compact tame sets on M. A functional $\mu : \mathcal{P}(M) \to \mathbb{R}$ is called a smooth valuation if it can be written as*

$$\mu(P) = \int_P \phi + \int_{\mathrm{Nor}(P)} \omega, \quad \phi \in \Omega^n(M), \omega \in \Omega^{n-1}(\mathrm{Gr}_{n-1}^+(TM)).$$

The space of smooth valuations is denoted by $\mathcal{V}(M)$. A functional $\Phi : \mathcal{P}(M) \times \mathfrak{B}(M) \to \mathbb{R}$ is called a smooth curvature measure if it can be written as

$$\Phi(P,U) = \int_{P \cap U} \phi + \int_{\mathrm{Nor}(P)\cap\pi^{-1}(U)} \omega, \quad \phi \in \Omega^n(M), \omega \in \Omega^{n-1}(\mathrm{Gr}_{n-1}^+(TM)).$$

The space of smooth curvature measures is denoted by $\mathcal{C}(M)$.

Proposition 2.3 ([**17**]). *The intrinsic volumes on a riemannian manifold are smooth valuations.*

Examples: $\mu_0^M(X) = \chi(X)$. Actually Chern in his intrinsic proof of the Chern-Gauss-Bonnet theorem [**15**] proves that χ is a smooth valuation (without using this terminology of course).

Given an isometric embedding $M \to N$ of manifolds, there is an obvious restriction map $\mathcal{V}(N) \to \mathcal{V}(M)$.

Theorem 2.4 (Enhanced Weyl principle).
- *If $(M,g) \to (\tilde{M},\tilde{g})$ is an isometric embedding, then $\mu_k^{\tilde{M}}|_M = \mu_k^M$.*
- *Conversely, let μ be a functor that associates to each riemannian manifold a smooth valuation ϕ^M such that $\phi^{\tilde{M}}|_M = \phi^M$ whenever $(M,g) \to (\tilde{M},\tilde{g})$ is an isometric embedding. Then there are constants c_k such that*

$$\mu = \sum_{k=0}^{\infty} c_k \mu_k.$$

3. The scalar curvature measure of an o-minimal set

The three main curvature quantities on a riemannian manifold are the sectional curvature, the Ricci curvature and the scalar curvature. There have been different attempts to generalize such quantities to certain singular sets. A well known theory is the metric approach to (lower and upper) sectional curvature bounds by Alexandrov, Toponogov, Burago, Gromov, Perelman and many others [**10, 12, 13**]. The key idea is to do compare triangles in a metric space with triangles in a model space of constant sectional curvature. Concerning Ricci curvature, the basic setting is that of a metric measure space, and (lower) Ricci curvature bounds are expressed using convexity properties of optimal mass transport (see for instance [**21, 28**]). Inspired by Weyl's principle, we use integral geometry to define a notion of scalar curvature measure on tame sets.

Recall that on an n-dimensional riemannian manifold, the $(n-2)$nd Lipschitz-Killing curvature measure is given by $\Phi_{n-2}(M,U) = \frac{1}{4\pi}\int_U s\,d\,\mathrm{vol}$. This suggests that $\mathrm{scal}(X,\bullet) := 4\pi\Phi_{n-2}(X,\bullet)$ may be a candidate for a scalar curvature measure on a tame set X.

Theorem 3.1 ([**3**]). *Let $X \subset \mathbb{R}^N$ be an n-dimensional semialgebraic set (or definable in some o-minimal structure) with a Nash stratification as in Theorem 1.7. Then for Borel subset $U \subset \mathbb{R}^N$*

$$\begin{aligned}\mathrm{scal}(X,U) = &\sum_{S\in\mathcal{S}_n}\int_{S\cap U} s(x)d\,\mathrm{vol}_n(x) + 2\sum_{S\in\mathcal{S}_{n-1}}\int_{S\cap U}\mathrm{tr}(II_x)d\,\mathrm{vol}_{n-1}(x)\\ &+ 4\pi\sum_{S\in\mathcal{S}_{n-2}}\int_{S\cap U}\left(\frac{1}{2}+(-1)^n\frac{\chi_{loc}(X,x)}{2}-\theta_n(x)\right)d\,\mathrm{vol}_{n-2}\end{aligned}$$

Here $\mathrm{tr}(II_x)$ is the sum of the traces of the fundamental forms at x of all n-dimensional strata that contain S in their boundary, $\chi_{loc}(X,x) = \chi(X,X\backslash\{x\})$ is the local Euler characteristic, which is constant along the stratum, and $\theta(x) := \lim_{r\to 0}\frac{\mathrm{vol}_n(X\cap B(x,r))}{\omega_n r^n}$ is the density of X at a point x.

On a riemannian manifold with sectional curvature $K \geqslant \kappa$ (or $K \leqslant \kappa$), the scalar curvature is bounded from below (or above) by $\kappa n(n-1)$. This can be generalized to semialgebraic sets in the following way.

A semialgebraic subset of $\mathbb{R}^N$ has the induced metric, but it has also an induced inner metric. The distance between two points is the infimum over the length of curves joining them.

Theorem 3.2 ([**3, 4**]).
(1) *Let X be a compact connected semialgebraic set which is an Alexandrov space with curvature bounded below by $\kappa \in \mathbb{R}$ for its inner metric. Then*

$$\mathrm{scal}(X,\bullet) \geqslant \kappa n(n-1)\,\mathrm{vol}\,.$$

(2) *Let X be a compact connected semialgebraic pseudo-manifold which is a space with curvature bounded from above by κ (i.e. locally a* $\mathrm{CAT}(\kappa)$*-space), then*

$$\mathrm{scal}(X,\bullet) \leqslant \kappa n(n-1)\,\mathrm{vol}\,.$$

The word pseudo-manifold means that there exists a triangulation with certain good properties that make the local topology at each point of S look like a manifold. In particular, the number of n-dimensional strata meeting in an $(n-1)$-dimensional stratum is always 2, and the local Euler characteristic at each point is 1. Finally, saying that one signed measure is smaller than another means that the difference is a positive measure.

On a riemannian manifold, a lower Ricci curvature bound $\mathrm{Ric} \geqslant \kappa g$ implies a lower scalar curvature bound $s \geqslant n\kappa$. It is not known whether a synthetic lower Ricci bound on a semialgebraic set implies a lower bound on the scalar curvature measure.

4. Hilbert-Einstein functional

Let (M, g) be a compact riemannian manifold of dimension n with scalar curvature s and volume form vol. Let $g_t, t \in (-\epsilon, \epsilon)$ be a variation of the metric, with $h := \frac{d}{dt}|_{t=0} g_t$, which is a symmetric bilinear form, not necessarily positive definite. Let s_t and vol_t be the scalar curvature and volume form of (M, g_t). Then

$$\frac{d}{dt}|_{t=0} \int_M s_t d\,\mathrm{vol}_t = \int_M \left\langle h, \mathrm{ric} - \frac{s}{2} g \right\rangle d\,\mathrm{vol}\,.$$

The tensor $E = \mathrm{ric} - \frac{s}{2} g$ is the Einstein tensor. We want to generalize this formula to tame sets. For this, let (M, g) be a riemannian manifold of dimension m and $X \subset M$ be a fixed tame set of dimension n. Let g_t be a variation of the riemannian metric as above. Since $\mu_{n-2}^M(X, \bullet)$ is up to a multiple a good candidate for the scalar curvature measure of X, we can consider $\frac{d}{dt}\big|_{t=0} \mu_{n-2}^{(M,g_t)}(X, \bullet)$.

Theorem 4.1 (Variations of intrinsic volumes, [**5**]). *For each riemannian manifold (M, g) and each compactly supported symmetric bilinear form h on M, there exists a smooth valuation $\mu_k^{(M,g),h}$ on M such that for all compact tame sets $X \subset M$.*

$$\frac{d}{dt}|_{t=0} \mu_k^{(M,g+th)}(X) = \mu_k^{(M,g),h}(X).$$

For fixed X, the map $h \mapsto \mu_k^{(M,g),h}(X)$ defines a generalized section of the vector bundle of symmetric bilinear forms on M (wait for the next lecture to see the definition of a generalized section).

For example, if X is n-dimensional and if the support of h is contained in an n-dimensional stratum S of X, then

$$\mu_{n-2}^{(M,g),h}(X) = \frac{1}{4\pi} \int_S \langle h, \mathrm{ric} - \frac{s}{2} g \rangle d\,\mathrm{vol}\,.$$

However, at lower dimensional strata, there are contributions that are not smooth symmetric bilinear forms, but generalized ones.

CHAPTER 3

Lecture III: Pseudo-riemannian case

1. A quick reminder on generalized objects

We follow [**1**]. Let X be a smooth manifold of dimension n. Let $C^\infty(X)$ be the Fréchet space of smooth complex valued functions on X. Let $\mathcal{M}^\infty(X)$ be the Fréchet space of smooth complex valued measures on X. By a subscript c we denote the elements with compact support. We then have a map

$$C^\infty(X) \times \mathcal{M}^\infty_c(X) \to \mathbb{C}, (f, \mu) \mapsto \int_X f\mu$$

that induces an injection with dense image

$$C^\infty(X) \hookrightarrow (\mathcal{M}^\infty_c(X))^* =: C^{-\infty}(X).$$

Elements of the space on the right hand side are called generalized functions. Similarly, there is a map

$$C^\infty_c(X) \times \mathcal{M}^\infty(X) \to \mathbb{C}, (f, \mu) \mapsto \int_X f\mu$$

that induces an injection with dense image

$$M^\infty(X) \hookrightarrow (C^\infty_c(X))^* =: M^{-\infty}(X).$$

Elements of the space on the right hand side are called generalized measures. For example, given $x \in X$, the Dirac delta $\delta_0(f) := f(0)$ is a generalized measure, not a generalized function. But we can also define a generalized function by setting $\delta_0(fdx) := f(0)$. Sometimes the word *distribution* is used for either of the two notions, but it may lead to confusion so we try to avoid it. In the case of $X = \mathbb{R}^n$, one can identify functions and measures by $f \mapsto fdx_1 \wedge \ldots \wedge dx_n$, but such an identification is not possible on a manifold without extra structure (such as riemannian metric).

In general, the product of two generalized functions is not well-defined. However, under some conditions on the singularities of the functions involved (in terms of microlocal analysis, wave front sets) it is possible to define their product. Similarly, we can not, in general, restrict a generalized function to a submanifold, but under some conditions it is possible.

More generally, let $\mathcal{E} \to X$ be a finite-dimensional vector bundle over X. Let $|\omega_X|$ be the line bundle of densities on X (the fiber at a point x is the space of Lebesgue measures on T_xX). Then we have a pairing

$$C^\infty(X, \mathcal{E}) \times C^\infty_c(X, \mathcal{E}^* \otimes |\omega_x|) \to \mathbb{R},$$

just pair an element of $\mathcal{E}_x$ with an element of $\mathcal{E}_x^*$, and integrate the remaining density over X. The induced map

$$C^\infty(X,\mathcal{E}) \hookrightarrow (C_c^\infty(X,\mathcal{E}^*\otimes|\omega_x|))^* =: C^{-\infty}(X,\mathcal{E})$$

is continuous, injective, and has dense image. Elements of the space on the right hand side are called generalized sections.

2. Homogeneous generalized functions on the real line

We refer to [**18**] and [**20**, Section 3.2] for the material in this section.

Recall that the Dirac function δ_0 satisfies

$$\langle \delta_0^{(j)}, f(x)dx\rangle = (-1)^j f^{(j)}(0).$$

Given $x \in \mathbb{R}$, we write $x = x_+ - x_-$ with $x_+ = \max(0,x), x_- = -\min(0,x)$. Let $s \in \mathbb{C}$ with $\mathrm{Re}\, s > -1$. Then the function x_+^s is locally integrable and defines a generalized function on $\mathbb{R}$. It is well-known that x_+^s extends to a meromorphic family of generalized functions with simple poles at $s = -1, -2, \ldots$. Explicitly, if $-k-1 < \mathrm{Re}\, s < -k, k \in \mathbb{N}$ and $\phi \in C_c^\infty(\mathbb{R})$, then

$$\langle x_\pm^s, \phi(x)dx\rangle = \int_0^\infty x^s \left(\phi(\pm x) - \sum_{i=0}^{k-1} (\pm 1)^i \frac{\phi^{(i)}(0)}{i!} x^i\right) dx, \tag{1}$$

and

$$\mathrm{Res}_{s=-k}\, x_\pm^s = (\mp 1)^{k-1} \frac{\delta_0^{(k-1)}}{(k-1)!}. \tag{2}$$

Similarly, the locally integrable functions given by $|x|^s := x_+^s + x_-^s, \mathrm{sign}(x)|x|^s := x_+^s - x_-^s$ for $\mathrm{Re}\, s > -1$, extend to meromorphic families of generalized functions. The poles and residues are given by

$$\mathrm{Res}_{s=-2k+1}\, |x|^s = 2\frac{\delta_0^{(2k-2)}}{(2k-2)!} \tag{3}$$

$$\mathrm{Res}_{s=-2k}\, \mathrm{sign}(x)|x|^s = -2\frac{\delta_0^{(2k-1)}}{(2k-1)!}. \tag{4}$$

3. Generalized valuations

A key feature of the space $\mathcal{V}(M)$ of smooth valuations on a manifold is the existence of a product structure satisfying a version of Poincaré duality. The formal definition/construction is involved and we will not give it here. To get an idea, we consider a special situation. Let

$$\phi_i(X) = \int_{\mathcal{P}(M)} \chi(X\cap Y_i)dm_i(Y_i), \quad i=1,2,$$

where m_i is a signed measure on $\mathcal{P}(M)$. Then (under some conditions), we have

$$\phi_1 \cdot \phi_2(X) = \int_{\mathcal{P}(M)\times\mathcal{P}(M)} \chi(X \cap Y_1 \cap Y_2) d(m_1 \times m_2)(Y_1, Y_2).$$

PROPOSITION 3.1. (1) *The product is commutative and associative.*
(2) *The Euler characteristic is the unit element.*
(3) *If M is riemannian, then the intrinsic volumes satisfy $\mu_k^M \cdot \mu_l^M = \mu_{k+l}^M$.*

Let $\mathcal{V}_c(M)$ be the space of smooth valuations with compact support. There is an obvious functional

$$\int_M : \mathcal{V}_c(M) \to \mathbb{R}, \phi \mapsto \phi(M).$$

(If M is non-compact, we are not really allowed to plug M into ϕ, but we can plug in some large part of M that contains the support of ϕ).

PROPOSITION 3.2. *The pairing*

$$\mathcal{V}(M) \times \mathcal{V}_c(M) \to \mathbb{R}, (\phi_1, \phi_2) \mapsto \int_M \phi_1 \cdot \phi_2$$

is perfect. The induced map

$$\mathcal{V}(M) \hookrightarrow \mathcal{V}_c(M)^* =: \mathcal{V}^{-\infty}(M)$$

is injective with dense image. The elements on the space on the right hand side are called generalized valuations.

Example: let $X \subset M$ be a tame subset. Then $\phi \mapsto \phi(X)$ is a generalized valuation, denoted by χ_X.

Analogous to the case of generalized functions, the product of two generalized valuations can only be defined if certain conditions are satisfied. Let X_1, X_2 be tame subsets that intersect transversally. What this means depends on the precise class of tame sets. If X_1, X_2 are submanifolds with corners, it is enough to assume that all pairs of strata intersect transversally. Then the product $\chi_{X_1} \cdot \chi_{X_2}$ is defined and by [**2**] we have

$$\chi_{X_1} \cdot \chi_{X_2} = \chi_{X_1 \cap X_2}.$$

Similarly, the restriction to a submanifold $M' \subset M$ is only defined under certain conditions on the generalized valuation. If X is a tame set that intersects M' transversally, then $\chi_X|_{M'} = \chi_{X \cap M'}$.

A general principle is that generalized valuations can not be evaluated on all tame sets, but only at particular ones. For instance, we have $\chi_X(Y) = \chi(X \cap Y)$ whenever X and Y are transversal, but otherwise $\chi_X(Y)$ is not defined.

4. Intrinsic volumes on pseudo-riemannian manifolds

By a pseudo-euclidean space we understand a finite-dimensional real vector space with a non-degenerate symmetric blinear form, that is not assumed to be positive definite. A pseudo-riemannian manifold is a smooth manifold with a smooth field of pseudo-euclidean scalar products on the tangent spaces. If M is connected, then the signature is constant on M and given by some (p, q). The case $q = 0$ is the riemannian case, the case $q = 1$ is the lorentzian case.

In the pseudo-euclidean/pseudo-riemannian setting, we try to mimic as much as we can from the theory of intrinsic volumes. There is no satisfying analogue of Steiner's formula, since tubes are in general non-compact due to the existence of null-vectors.

We first have a version of Hadwiger's theorem.

THEOREM 4.1. *Let V be a peudo-euclidean vector space of dimension $n = p + q$ with signature (p, q) and isometry group $\mathrm{O}(p, q)$. For $k \in \{0, n\}$,*

$$\dim \mathrm{Val}_k^{-\infty}(V)^{\mathrm{O}(p,q)} = 1;$$

and for $1 \leqslant k \leqslant n-1$,

$$\dim \mathrm{Val}_k^{-\infty}(V)^{\mathrm{O}(p,q)} = \begin{cases} 1 & \min(p,q) = 0, \\ 2 & \min(p,q) \geqslant 1. \end{cases}$$

Except in some special cases, these valuations are not continuous. It is convenient to think of the two valuations in each degree as being real and imaginary part of a single complex-valued valuation.

Let us comment on the construction of these generalized valuations. We start with the euclidean case. Recall from Definition 2.3 that a smooth valuation μ is given by a pair of differential forms (ϕ, ω), where $\phi \in \Omega^n(V)$ and $\omega \in \Omega^{n-1}(\mathrm{Gr}_{n-1}^+(TV))$. If μ is translation-invariant and $\mathrm{SO}(n)$-invariant, then these forms can be chosen to be invariant as well. The group generated by translations and rotations acts transitively on $\mathrm{Gr}_{n-1}^+(TV)$. The space of invariant forms is therefore easy to describe. Let us now turn to the pseudo-euclidean case. A generalized valuation is given by a pair (ϕ, ω) of generalized forms. They can be thought of as differential forms whose coefficients are generalized functions. A major problem is that the group generated by translations and $\mathrm{O}(p, q)$ does not act transitively on $\mathrm{Gr}_{n-1}^+(TV)$. There are typically two open orbits of time and space like hyperplanes, and one orbit of light like planes. On each open orbit, one can easily write down all invariant differential forms, and they are smooth where they are defined. However, to get a generalized form on the whole space, they have to be glued together along the closed orbit. A rough analogy is with generalized functions on the real line: on $(-\infty, 0)$, we have the smooth function $f_-(x) := x_-^s$, and on $(0, \infty)$ we have the smooth function $f_+(x) := x_+^s$. Some linear combinations can be glued together to give a generalized function on $\mathbb{R}$, sometimes one has to pass to residues, see Section 2.

Next, we prove a version of Weyl's principle.

THEOREM 4.2 (Weyl principle in the pseudo-riemannian category). *For every pseudo-riemannian manifold* (M, g)*, there are complex-valued intrinsic volumes* μ_k^M*, which are generalized valuations, such that whenever* $(M, g) \to (\tilde{M}, \tilde{g})$ *is an isometric embedding of pseudo-riemannian manifolds, then* $\mu_k^{\tilde{M}}|_M = \mu_k^M$*. In particular the restriction is well-defined. Conversely, if we have a family* μ^M *of generalized valuations on pseudo-riemannian manifolds (i.e. one generalized valuation on* M *for each pseudo-riemannian manifold* (M, g)*) such that* $\mu^{\tilde{M}}|_M = \mu^M$ *for each isometric embedding, then* μ *is given by a unique infinite linear combination* $\mu = \sum_{k=0}^{\infty} a_k \mu_k + \sum_{k=1}^{\infty} b_k \bar{\mu}_k$.

The construction is again by finding suitable generalized forms, but the curvature of the manifold (M, g) has to be taken into account.

PROPOSITION 4.3 (Basic properties of μ_k, [**7**]).

(1) *If* (M, g) *is riemannian, then* μ_k^M *is the usual* k*-th intrinsic volume.*
(2) $\mu_k^{M,g}$ *depend continuously on* g *in the* C^∞ *topology.*
(3) *Homogeneity:*
$$\mu_k^{X,\lambda g} = \begin{cases} \sqrt{\lambda}^k \mu_k^{X,g}, & \lambda > 0 \\ \sqrt{\lambda}^k \overline{\mu}_k^{X,g}, & \lambda < 0 \end{cases}$$
(4) $\mu_0^M = \chi$.

PROPOSITION 4.4 (Künneth-type formula, [**6**]). *Let* (M_i, g_i) *be pseudo-riemannian manifolds and* $X_i \subset M_i$ *tame that satisfy a certain transversality condition (see Faifman's talk). Then*
$$\mu_k^{M_1 \times M_2}(X_1 \times X_2) = \sum_{k_1+k_2=k} \mu_{k_1}^{M_1}(X_1) \times \mu_{k_2}^{M_2}(X_2).$$

In the riemannian case, we can always restrict the intrinsic volumes to submanifolds, since they are riemannian manifolds themselves. In the pseudo-riemannian case, this is of course not true anymore: a submanifold of a pseudo-riemannian manifold is in general not a pseudo-riemannian manifold. Nevertheless, the intrinsic volumes from Theorem 4.2 can be restricted to a class of submanifolds called LC-regular. This will be the topic of D. Faifman's talk.

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