The Grothendieck conjecture for affine curves

Oberseminar Arithmetische Homotopietheorie

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Introduction

Grothendieck's "anabelian geometry", as first laid out in his letter [Gro83] to Faltings, is concerned with the question to which extent arithmetic-geometric properties of a variety X over a field k are determined by the exact sequence of profinite groups

$$1 \to \bar{\pi} \to \pi \to G \to 1,$$

where $\pi = \pi_1^{\text{ét}}(X, \bar{x})$ and $\bar{\pi} = \pi_1^{\text{ét}}(\bar{X}, \bar{x})$ denote the étale fundamental group of X and the base change $\bar{X} = X \otimes_k \bar{k}$, respectively, and where $G = \mathrm{Gal}(\bar{k}/k)$ denotes the absolute Galois group of k.

Among other things, he conjectured the following:

Conjecture. Let X be a hyperbolic curve over a field k that is finitely generated over $\mathbb{Q}$. Then the isomorphism class of $\pi \twoheadrightarrow G$ uniquely determines the isomorphism class of X/k.

In his doctoral thesis [Tam97], Tamagawa managed to prove an even stronger form of the above conjecture for *affine* hyperbolic curves:

Theorem (Tamagawa '97, [Tam97, Theorem 0.3])**.** Let k be a field finitely generated over $\mathbb{Q}$. Let X and X' be affine hyperbolic curves over k with étale fundamental groups π and π' respectively. Then the canonical map of sets

$$\mathrm{Isom}_k(X, X') \xrightarrow{\sim} \mathrm{Isom}_G^{\mathrm{out}}(\pi, \pi'),$$

where $\mathrm{Isom}_G^{\mathrm{out}}(\pi, \pi')$ denotes the set of *outer* G-isomorphisms, is bijective.

Astonishingly, Tamagawa proves his theorem by first proving a finite field version [Tam97, Theorem (0.5)] (something that Grothendieck didn't even conjecture to be true!) and then reducing the statement over fields finitely generated over $\mathbb{Q}$ to this case.

The goal of this seminar is to work through Tamagawa's thesis [Tam97] and the above two theorems [Tam97, Theorem (0.5)] and [Tam97, Theorem (0.3)] in detail.

Literature

The main source is Tamagawa's original paper [Tam97] but it might be handy to have [BLR00, §12, §13] close by. For some of the topics, there are some further helpful references. These are listed in the relevant paragraph.

Time and Place

- We'll meet in person every two weeks on *Wednesday* in either Heidelberg (SR *tba*) or Frankfurt (Hörsaal 16).
- After the session at *08.06.*, there will be a one-time shift of a week, which means the next session will be at *29.06.*.
- Usually, a session will consist of two talks:
  - Heidelberg
    The first talk will be from 11 : 30 - 13 : 00 and the second one from 14 : 15 - 15 : 45.
  - Frankfurt
    The first talk will be from 12 : 15 - 13 : 45 and the second one from 14 : 30 - 16 : 00.

  Ideally, each talk is prepared with about 10-15 *minutes for discussion in mind.*
- Talks 7—9 will be held in one longer session, details *tba.*
- After the last talk, we'll hike along the philosophers' path and afterwards have dinner together at *tba.*

Schedule

Overview *(Online, 20.04.)*

0. Provide an overview of the seminar and distribute the topics.

Generalities on the fundamental groups of curves *(Frankfurt, 27.04.)*

[BLR00, §12, §13], [AM69, §3]

1. Introduce the setup of [Tam97, §1] (up to [Tam97, (1.1)]). Define the notion of a "full" class $\mathfrak{C}$ of finite groups and define the maximal pro-$\mathfrak{C}$ quotient $G^{\mathfrak{C}}$ of a profinite group G. Explain [Tam97, (1.1)] and its corollaries [Tam97, (1.2), (1.4)].

2. Discuss torsion-, and centrefreeness of fundamental groups of curves:
Prove [Tam97, (1.5)] and deduce that the pro-$\mathfrak{C}$ quotient $\bar{\pi}^{\mathfrak{C}}$ of the geometric fundamental group $\bar{\pi}$ of a curve is torsionfree [Tam97, (1.6)]. Prove that $\bar{\pi}^{\mathfrak{C}}$ is centrefree [Tam97, (1.11)] (a more accessible proof can be found in [Col98, Lemma 1]).

Characterization of Decomposition Groups *(Heidelberg, 11.05.)*

[Col98, Theorem 3], [Sti13, §4], [Sti13, Proposition 54]

3. Define (geometric) (quasi-)sections [Tam97, (2.3)] and the property “A” of a field k [Tam97, p. 150]. Explain [Tam97, (2.6)] (see [Sti12, Lemma 7] for a more elaborate version) and use it to prove the first item of the main result [Tam97, (2.8)] regarding the characterization of decomposition groups.

4. Prove the second item of [Tam97, (2.8)]. Define the properties “B” and “C” of a field k [Tam97, p. 150]. Discuss and, time permitting, prove the remaining items of [Tam97, (2.8)] (also see [Col98, Theorem 3]). Focus on accessibility rather than completeness.

Characterization of various invariants *(Frankfurt, 25.05.)*

[BLR00, §12.3]

5. Explain how to group-theoretically characterize the characteristic $p \geq 0$ of the ground field [Tam97, (3.1)] as well as the geometric fundamental group $\bar{\pi}$ over finite fields [Tam97, (3.3)]. Prove the third item of [Tam97, (3.4)]. State and, time permitting, sketch how to characterize the geometric fundamental group $\bar{\pi}$ over fields finitely generated over $\mathbb{Q}$ [Tam97, (3.2)].

6. Explain how to group-theoretically recover the cardinality $q = \#k$ of the ground field and the Frobenius element φ_k (for k finite) (the first two items of [Tam97, (3.4)]). Prove how to recover the genus g of the curve as well as the number n of points of the boundary [Tam97, (3.5)]. State and, time permitting, prove how to recover the number of rational points [Tam97, (3.8)] of the curve as well as the kernel “I(π)” [Tam97, (3.7)].

The Grothendieck conjecture for curves over finite fields *(Heidelberg, 08.06.)*

The goal of this section is to prove the tame case of [Tam97, (4.3)].

7. Introduce the setup considered in [Tam97, §4] and elucidate the general strategy of the proof of (the tame case of) [Tam97, (4.3)]. Prove [Tam97, (4.1)]. Quickly introduce class field theory of function fields and explain how, together with the characterizations of the preceding talks, it can be applied to obtain a multiplicative isomorphism $\varphi^\times : K_1'^\times \xrightarrow{\sim} K_2'^\times$, where K_i' denotes the maximal pro-$\mathfrak{C}$ Galois extension of the composite $K_i \cdot k_i^{\mathrm{sep}}$ of the function field of the curve and the separable closure of the ground field under consideration, respectively ([Tam97, (4.3)] up to, including, [Tam97, (4.5)]).

8. Assuming the critical claim [Tam97, (4.6)] that the isomorphism $\varphi^\times$ is additive, finish the proof of [Tam97, (4.3)]. State [Tam97, (4.7)] and explain why it is applicable in the situation at hand. Prove [Tam97, (4.16)] and mention that we’ll eventually reduce [Tam97, (4.7)] to it.

9. Finish the proof of the main result [Tam97, (4.3)] over finite fields by proving [Tam97, (4.7)]:

Define *minimal elements* in function fields [Tam97, (4.8)], state and, time permitting, sketch [Tam97, (4.11)] and use it to prove that "$(a)(b)(c) \implies (a)(b)(c')$" [Tam97, p. 171]. Prove [Tam97, (4.13), (4.15)] and explain how this reduces the missing claim "$(a)(b)(c') \implies$ additivity of $\varphi^\times$" [Tam97, p. 172] to the already proven [Tam97, (4.16)].

"Anabelian" criterion for good reduction *(Frankfurt, **29.06.(!)**)*

10. Introduce the setup of [Tam97, §5]. Define the notion of having *good reduction* for tuples (X, D) consisting of a curve X and an effective étale divisor D at a point s of the base scheme S [Tam97, (5.1)]. Prove Tamagawa's "anabelian" criterion [Tam97, (5.3)] for detecting whether such a tuple (X, D) has good reduction in terms of $\pi_1^{\text{ét}}(X \setminus D)$.

11. Prove the key lemma [Tam97, (5.5)] and use it and Tamagawa's anabelian criterion for good reduction to group-theoretically recover the fundamental group of the special fibre of a curve over a DVR from the fundamental group of the curve itself [Tam97, (5.7)].

The Grothendieck conjecture for curves over fields finitely generated over $\mathbb{Q}$

(Heidelberg, 13.07.)

12. Reduce the main result in characteristic 0 [Tam97, (6.3)] to [Tam97, (6.6)].

13. Complete the proof of [Tam97, (6.3)] by proving the remaining claim [Tam97, (6.6)].

14. Hike along the philosophers' path with dinner at *tba*.

References

[AM69] Michael Artin and Berry Mazur. *Étale Homotopy Theory*. Lecture Notes in Mathematics. Springer-Verlag, 1969.

[BLR00] J. B. Bost, François Loeser, and Michel Max Raynaud. *Courbes semi-stables et groupe fondamental en géométrie algébrique : Luminy, décembre 1998*. 2000.

[Col98] Collectif. "Curves and their fundamental groups". en. In: *Séminaire Bourbaki : volume 1997/98, exposés 835-849*. Astérisque 252. talk:840. Société mathématique de France, 1998. URL: http://www.numdam.org/item/SB_1997-1998__40__131_0/.

[Gro83] Alexander Grothendieck. *Brief an Faltings*. 1983. URL: https://webusers.imj-prg.fr/~leila.schneps/grothendieckcircle/Letters/GanF.pdf.

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[Sti13] Jakob Stix. *Rational Points and Arithmetic of Fundamental Groups*. 2013. URL: https://link.springer.com/book/10.1007/978-3-642-30674-7.

[Tam97] Akio Tamagawa. "The Grothendieck conjecture for affine curves". In: *Compositio Mathematica* 109.2 (1997), pp. 135–194. DOI: 10.1023/A:1000114400142.