Die Dualität des Geldes

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Vienna in the 1930's:

Karl Menger founds the “Mathematische Kolloquium”. Among the participants:

- Kurt Gödel
- Abraham Wald
- Franz Alt
- and several other geniuses.

Triggered by questions of Karl Schlesinger and Oskar Morgenstern, the seminar also analyzed economic problems.

Question:

In a competitive market, do we always find price systems such that markets clear?

Previous work: LéonWalras, Vilfredo Pareto.

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What coherence requirements of the agents' preferences are necessary for such equilibria to exist?

Franz Alt (1936): "On the measurability of utility"

Abraham Wald (1938): the lost paper!
Under mild conditions there is an equilibrium system of prices.
Element of proof: A fixed point theorem.

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Mathematical Finance

Louis Bachelier (1900) "Théorie de la spéculation":

Models the price of a stock $(S_t)_{0 \leq t \leq T}$ in *continuous time* as a *stochastic process*.
Independently Albert Einstein (1905) and Marjan Smoluchowski (1906) use a similar model $(X_t)_{0 \leq t \leq T}$ to model the movement of a small particle in $\mathbb{R}^3$: "Brownian motion" (Robert Brown 1827).

Heuristics of Brownian motion:

$$dS_t := S_{t+dt} + S_t = \varepsilon_t \; dt^{\frac{1}{2}}$$

where $\mathbb{P}[\varepsilon_t = 1] = \mathbb{P}[\varepsilon_t = -1] = \frac{1}{2}$

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An OLD MESSAGE FROM LUCRECIUS 99-55BC

DE RERUM NATURA LIBER SECUNDUS 125 - 141
Hoc etiam magis haec animum te advertere par est
corpora quae in solis radiis turbare videntur,
quod tales turbae motus quoque materiai
significant clandestinos caecosque subesse.
Multa videbis enim plagis ibi percita caecis
commutare viam retroque repulsa reverti
nunc huc nunc illuc in cunctas undique partis.
Scilicet hic a principiis est omnibus error.
Prima moventur enim per se primordia rerum,
inde ea quae parvo sunt corpora conciliatu
et quasi proxima sunt ad viris principiorum,
ictibus illorum caecis inpulsa cientur,
ipsaque $<$ pro $>$ porro paulo maiora lacessunt.
Sic a principiis ascendit motus et exit
paulatim nostros ad sensus, ut moveantur
illa quoque, in solis quae lumine cernere quimus
nec quibus id faciant plagis apparet aperte.

Deutsche Übersetzung

Noch stärker als das gerade Besprochene solltest du folgendes
zur Kenntnis nehmen:
Dass die Körperchen, die man im Sonnenstrahl herumtanzen
sieht,
signalisieren, dass auch der Materie solche konfusen
Bewegungen innewohnen,
und zwar unsichtbar und blind.

Denn du wirst begreifen, dass Vieles dort, von blinden Stössen
in Bewegung gesetzt,
seine Richtung ändert und auf einen Stoss hin sich wieder in
die Gegenrichtung bewegt,
bald hierhin und bald dorthin, überallhin,
in alle Richtungen:
Und dieses Herumirren (der Teilchen) gibt es vom Anfang an,
von den Atomen her.

Zunächst nämlich bewegen sich die Ursprünge der Dinge
(= die Atome) an sich;
daraufhin werden die Körper, die nur aus wenigen Atomen
zusammengebacken sind,
und die quasi den Kräften der Atome am nächsten stehen,
von deren blinden Impulsen
in Bewegung versetzt und setzen ihrerseits wiederum
in Bewegung, was ein bisschen grösser ist.

So steigt die Bewegung von den Ursprüngen (= Atomen) her
auf und gelangt
allmählich in den Bereich unserer Sinneswahrnehmung,
sodass sich auch die Teilchen bewegen,
die wir im Sonnenlicht beobachten können,
ohne dass doch so ohne weiteres
zu erkennen wäre, auf welche Impulse hin sie das tun.

Übersetzung: Dr. Gottfried Kreuz, Univ. Konstanz

Bachelier's goal:

a rational theory of option pricing.

Bachelier's Fundamental Principle:

"L' éspérance mathématique du spéculateur est nul"
[The speculator *in average* does neither win nor lose].
Corresponds to the *market efficiency hypothesis in its strong form*.

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Consequence of Bachelier's principle:

The pricing rule for a *derivative security* paying the random amount C_T at time T is given by

$$C_0 := \mathbb{E}[C_T] = \int_\Omega C_T(\omega)\, d\mathbb{P}(\omega).$$

Example:

A European call-option pays the random amount

$$C_T = (S_T - K)_+,$$

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Following Bachelier's pioneering work:

Paul Samuelson (1965)

Robert Merton (1973), Fisher Black, Myron Scholes (1973)

⋮

The Black-Scholes model: Brownian motion with drift on an exponential scale.

⋮

Replace Bachelier's fundamental principle by the **no arbitrage principle**.

Modelization of trading:

For a *trading strategy* $(H_t)_{0 \leq t \leq T}$ on the stock $(S_t)_{0 \leq t \leq T}$ the *total gain* at time T is the random variable

$$X_T = \int_0^T H_t \, dS_t$$

Motivation:

$$X_{t+dt} - X_t = H_t(S_{t+dt} - S_t).$$

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Definition:

An **arbitrage** for the stock $(S_t)_{0 \leq t \leq T}$ is a trading strategy $(H_t)_{0 \leq t \leq T}$ such that

$$\mathbb{P}[X_T = \int_0^T H_t dS_t \geq 0] = 1 \text{ and } \mathbb{P}[X_T = \int_0^T H_t dS_t > 0] > 0.$$

Economically convincing argument:

In a liquid financial market there is no arbitrage.

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Fundamental Theorem of Asset Pricing:

If $(S_t)_{0 \leq t \leq T}$ does not allow for an arbitrage, we can change the probability measure $\mathbb{P}$ to an equivalent probability measure $\mathbb{Q}$ such that Bachelier's principle holds true for $(S_t)_{0 \leq t \leq T}$ under the measure $\mathbb{Q}$.
$\mathbb{Q}$ is called an "equivalent risk neutral measure" or an "equivalent martingale measure".

Challenge:

Put this "meta-theorem" into a precise mathematical form and prove it rigorously!
Ross (1976), Harrison-Kreps (1979), Harrison-Pliska (1981), Kreps (1981),
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Consequence (via duality):

For a contingent claim C_T the arbitrage free prices C_0 at time $t = 0$ are given by the interval

$$\left(\inf_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}[C_T], \quad \sup_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}[C_T] \right),$$

where $\mathbb{Q}$ runs through the equivalent martingale measures.

Proof:

duality theory of infinite-dimensional vector spaces.

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Proof:

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Special case:

When there is only one equivalent martingale measure $\mathbb{Q}$ for the model $(S_t)_{0 \leq t \leq T}$, the interval reduces to one single price

$$C_0 = \mathbb{E}_{\mathbb{Q}}[C_T].$$

In this case we can *replicate* the contingent claim C_T by a trading strategy $(H_t)_{0 \leq t \leq T}$

$$C_T = C_0 + \int_0^T H_t dS_t$$

In the Bachelier as well as in the Black-Scholes model there is precisely one equivalent martingale measure $\mathbb{Q}$ ("complete markets").

But:

These models, based on the Gaussian distribution, underestimate the probability of extreme events!

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Portfolio Optimization:

How should an economic agent trade optimally?

Ingredients:

- initial endowment x
- financial market given by one (or many) stocks $(S_t)_{0 \leq t \leq T}$.
- preferences of the agent modeled by a utility function $U : \mathbb{R}_+ \to \mathbb{R}$, e.g. $U(x) = \log(x)$.

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Harry Markowitz (1953):

one period model, mean variance analysis (corresponds to a quadratic utility function U)

Robert Merton (1973):

portfolio optimization in continuous time.

$$(P) \quad \mathbb{E}_{\mathbb{P}}\left[U(x + \int_0^T H_t \, dS_t)\right] \to max!$$

where H runs through all (*admissible*) trading strategies $(H_t)_{0 \leq t \leq T}$.

Question:

Does the optimizer $(\hat{H}_t)_{0 \leq t \leq T}$ exist and how to calculate it?

"Primal" methods:

partial differential equations and variational analysis (Hamilton-Jacobi-Bellman).

"Dual" methods:

(Bismut (1973), Pliska (1986), He-Pearson (1989), Karatzas-Lehoczky-Shreve (1989)): optimization over the "dual" objects, i.e., the equivalent martingale measures.

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Motivation:

Suppose first that $(S_t)_{0\leq t\leq T}$ defines a *complete market* so that there is a *unique equivalent martingale measure* $\mathbb{Q}$.

If $(\hat{H}_t)_{0\leq t\leq T}$ is the optimal trading strategy and

$$\hat{X}_T = x + \int_0^T \hat{H}_t \, dS_t$$

denotes the corresponding optimal wealth at time T, then the following relation is – at least – economically appealing

$$(R) \quad U'(\hat{X}_T(\omega)) = const \frac{d\mathbb{Q}(\omega)}{d\mathbb{P}(\omega)}, \quad \omega \in \Omega,$$

where the constant depends on the initial endowment x, *but not on* $\omega \in \Omega$.

Admitting the relation (R) we can derive $\hat{X}_T$ and therefore also the optimal trading strategy $(\hat{H}_t)_{0\leq t\leq T}$ from $\frac{d\mathbb{Q}}{d\mathbb{P}}$.

Challenge:

Give a precise mathematical framework (as general as possible) and prove rigorously that things really work a they should.

Question:

What are the precise regularity requirements?

Answer (Kramkov-Schachermayer (1999):

We do not need any special regularity requirements on the stock price process $(S_t)_{0 \leq t \leq T}$. On the utility function U we need the *asymptotic elasticity conditon*.

$$\limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1.$$

Good guys: $U(x) = \log(x)$, $U(x) = \frac{x^p}{p}$ with $p < 1$

Bad guy: $U(x) \sim \frac{x}{\log(x)}$

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Beyond semi-martingales

Recall that Brownian motion was defined (heuristically) via the *quadratic variation*

$$dS_t = \varepsilon_t \ dt^{\frac{1}{2}}$$

More generally, a (continuous) *semi-martingale* always satisfies

$$|dS_t| \sim dt^{\frac{1}{2}}$$

Mandelbrot (1961,...) and others propose *fractional Brownian motion* as models for stock prices.

$$|dS_t| \sim dt^h,$$

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Theorem (Delbaen-Schachermayer 1994):

If the (continuous) stochastic process $(S_t)_{0 \leq t \leq T}$ fails to be a semi-martingale, then it allows for arbitrage.

Trading under transaction costs

For a trading strategy $(H_t)_{0 \le t \le T}$ and transaction costs $\varepsilon > 0$, we again define the gain (or loss) as the random variable

$$X_T = \int_0^T H_t dS_t - \varepsilon \int_0^T S_t d\mathrm{Var}_t(H)$$

Arbitrage under transaction costs

We say that the model $(S_t)_{0 \le t \le T}$ of a financial market allows for arbitrage under ε transaction costs, if there is X_T as above s.t.

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Theorem (Guasoni-Rasonyi-Schachermayer 2008):

In a large class of models, *including fractional Brownian motion,* there is no arbitrage, for any given level $\varepsilon > 0$ of transaction costs.

In addition, for any given $\varepsilon > 0$, we can always find *consistent price systems*.

Remark

Adding transaction costs re-establishes the no arbitrage theory also for non semi-martingales.
However, the Black-Scholes pradigm of *replication* now fails dramatically (Soner-Shreve-Cvitanic 1995, Levental-Skorohod 1996).

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What to do?

- Utility maximization (portfolio optimization) does make good sense also in the presence of transaction costs:

$$u(x) = \sup_H \mathbb{E}\left[U\left(x + \int_0^T H_t \, dS_t - \varepsilon \int_0^T S_t \, d\mathrm{Var}_t(H)\right)\right].$$

  where $U(x)$ is a fixed concave, increasing function (e.g. $U(x) = \log(x)$.)

- This problem still makes sense for "random endowment" C_T

$$u(C_T) = \sup_H \mathbb{E}\left[U\left(C_T + \int_0^T H_t \, dS_t - \varepsilon \int_0^T S_t \, d\mathrm{Var}_t(H)\right)\right]$$

- Utility indifference pricing (de Finetti: "certainty equivalent"): define the price x for X_T implicitly by

$$u(x) = u(C_T)$$

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- Utility indifference pricing (de Finetti: "certainty equivalent"): define the price x for X_T implicitly by

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What to do?

- Utility maximization (portfolio optimization) does make good sense also in the presence of transaction costs:

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What to do?

- Let $\hat{H}^x$ and $\hat{H}^{C_T}$ be the optimizing strategies corresponding to x and C_T; the difference $\hat{H}^{C_T} - \hat{H}^x$ may be interpreted as a hedging strategy for C_T
- Research programm:
  derive an asymptotic expansion for $\varepsilon \to 0$ and $h \to \frac{1}{2}$ how the option prices and hedging strategies deviate from the classical Black-Scholes price (compare Fouque-Papanicolao-Sircar, Janecek-Shreve, Kramkov-Sirbu etc.).

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