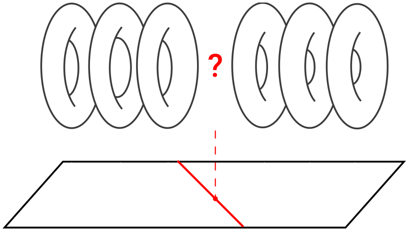
HITS
Heidelberg Institute for
Theoretical Studies

UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386

Thesis Defense

Singular fibers of the Hitchin fibration

Johannes Horn



Supervision: Daniele Alessandrini, Anna Wienhard

Content

Functions, just like living beings, are characterised by their singularities.

Paul Montel 1932

1. Hitchin Systems: Introduction and Motivation
2. Singular Fibers: Semi-abelian Spectral Data and Application

Higgs bundle moduli spaces

Riemann surface
genus $g \geq 2$

X

$\longrightarrow$ $\mathcal{M}_G(X)$

Moduli space of
G-Higgs bundles
on X

complex reductive
Lie group

G

$\mathrm{GL}(n, \mathbb{C}), \mathrm{Sp}(2n, \mathbb{C}), \ldots$

Properties of $\mathcal{M}_G(X)$:

i) Quasi-projective analytic space with hyperkähler structure on smooth points.

ii) Parametrizes solutions to geometric PDE: Hitchin equation.

iii) Non-abelian Hodge Theory: $\mathcal{M}_G(X) \cong \chi_G(\pi_1(X))$ diffeomorphic.

iv) Moduli space of vacuum states of some supersymmetric field theory.

v) **Dense subset fibered by Lagrangian tori, ”Hitchin system”.**

Hitchin system

Hitchin map:

$$\mathrm{Hit}_G : \mathcal{M}_G(X) \to B_G(X)$$

- Hitchin base: $B_G(X)$ $\mathbb{C}$-vector spaces, $\dim B_G(X) = \frac{1}{2} \dim \mathcal{M}_G(X)$.
- Hit_G is proper, surjective morphism.
- Hit_G defines a half-dimensional system of Poisson-commuting functions.

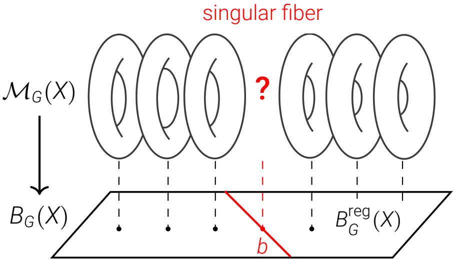
Theorem (Hi87, Sc96)

There exists a dense subset $B_G^{\mathrm{reg}} \subset B_G$, such that

$$\mathrm{Hit}_G : \mathrm{Hit}_G^{-1}(B_G^{\mathrm{reg}}) \to B_G^{\mathrm{reg}}$$

is a fibration by algebraic tori

⇒ algebraically completely integrable system.



Focus of present work: Singular fibers of the Hitchin system.

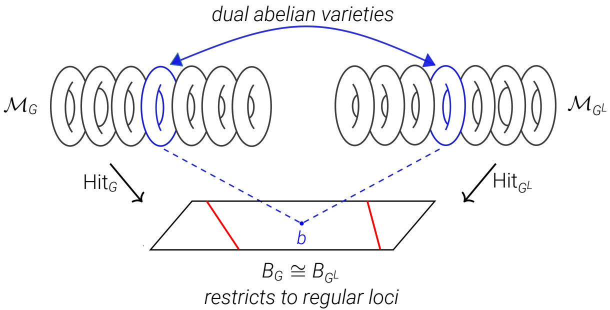
Langlands duality of Hitchin systems

G complex reductive Lie group $\implies$ Langlands dual group G^L.

Examples: $\mathrm{SL}(n,\mathbb{C})^L = \mathrm{PGL}(n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})^L = \mathrm{SO}(2n+1,\mathbb{C})$

Theorem (HaTh;Hi;DoPa)

The Hitchin systems of $\mathcal{M}_G(X)$ and $\mathcal{M}_{G^L}(X)$ are related as follows:



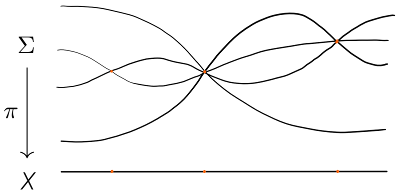
Reincarnation of mirror symmetry for Higgs bundle moduli spaces

Spectral Data

GL$(n, \mathbb{C})$-**Higgs bundle** (E, Φ):

i) E holomorphic vector bundle,

ii) Φ holomorphic matrix-valued one-form.



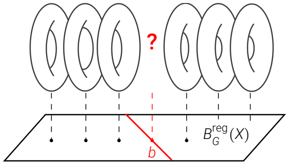
Spectral data / Abelianisation:

- Hitchin map fixes the eigenvalues of Φ $\iff$ spectral curve Σ.
- The Hitchin fibers parametrize the eigenspaces $\iff$ (generalised) line bundles on Σ.

Regular locus: $b \in B_G^{\mathrm{reg}}$ if and only if Σ is smooth.

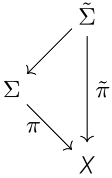
In this case: moduli space of eigen line bundles is an abelian torsor.
$=$ complex Lagrangian tori forming the integrable system.

What about the singular locus? How do the singular fibers look like?



· Singular Hitchin fibers

Semi-abelian spectral data



In general, the spectral curve Σ is **singular** at the ramification points.

Normalisation: Smoothing of complex algebraic curves.
$\tilde{\Sigma}$ = normalised spectral curve.

On $\tilde{\Sigma}$: (E, Φ) defines eigen line bundle $L \in \mathcal{A}_G(\tilde{\Sigma})$ abelian torsor.

But: L fixed by local deformations of (E, Φ) at singularities of Σ (Hecke transformations)

For now on: $G = \mathrm{Sp}(2n, \mathbb{C})$ or $G = \mathrm{SO}(2n+1, \mathbb{C})$

+ class of $\mathfrak{sl}(2)$-type Hitchin fibers:

- $G = \mathrm{Sp}(2, \mathbb{C}), \mathrm{SO}(3, \mathbb{C})$ all Hitchin fibers are of $\mathfrak{sl}(2)$-type.

Theorem (Stratification)

Let $\mathrm{Hit}_G^{-1}(b)$ of $\mathfrak{sl}(2)$-type, such that $\tilde{\pi} : \tilde{\Sigma} \to X$ is not unbranched. Then there exists a stratification

$$\mathrm{Hit}_G^{-1}(b) = \bigcup_{i \in I} \mathcal{S}_i$$

by locally closed subsets. Every stratum is isomorphic to a fiber bundle

$$\mathrm{Heck}_i := (\mathbb{C}^\times)^{r_i} \times \mathbb{C}^{s_i} \to \mathcal{S}_i \to \mathcal{A}_G(\tilde{\Sigma}).$$

Examples of singular Hitchin fibers

i) Two strata:
$\mathcal{S}_0$: Heck$_0 \cong \mathbb{C}$
$\mathcal{S}_1$: Heck$_1 \cong pt$

For $SL(2, \mathbb{C})$: Quadratic differential with single zero of order 3, all other zeroes simple.

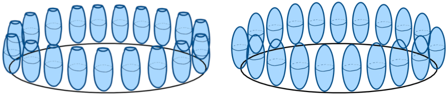


Figure: $\mathrm{Hit}_G^{-1}(b) \cong \mathbb{C}P^1$-bundle over abelian torsor

ii) Two strata:
$\mathcal{S}_0$: Heck$_0 \cong \mathbb{C}^\times$
$\mathcal{S}_1$: Heck$_1 \cong pt$

For $SL(2, \mathbb{C})$: Quadratic differential with single zero of order 2, all other zeroes simple.

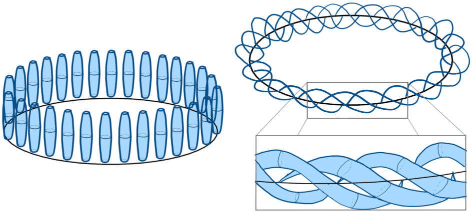


Figure: Twisted $\mathbb{C}P^1$-bundle over abelian torsor

Global description of $\mathfrak{sl}(2)$-type fibers

Strategy to obtain global picture of singular Hitchin fibers:

1. Compactify Heck_0 of open and dense stratum $\mathcal{S}_0$.
2. $\overline{\mathrm{Heck}_0}^{\mathrm{comp}} \to F_G(b) \to \mathcal{A}_G(\tilde{\Sigma})$ one-sheeted analytic covering of $\mathrm{Hit}_G^{-1}(b)$.

Example

i + ii) $\overline{\mathrm{Heck}_0}^{\mathrm{comp}} = \mathbb{C}P^1$

iii)

$$\text{Three strata:} \quad \begin{array}{ll} \mathcal{S}_0 : & \mathrm{Heck}_0 \cong \mathbb{C}^2 \\ \mathcal{S}_1 : & \mathrm{Heck}_1 \cong \mathbb{C} \\ \mathcal{S}_2 : & \mathrm{Heck}_2 \cong pt \end{array}$$

Up to normalisation:

$$\mathbb{P}(1,1,2) \to \mathrm{Hit}_G^{-1}(b_3) \to \mathcal{A}_G(b)$$

For $\mathrm{SL}(2,\mathbb{C})$: Quadratic differential with single zero order 4 or 5, all other zeroes simple.

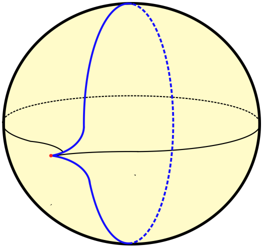


Figure: Stratification of $\mathbb{P}(1,1,2)$

App. 1: Langlands correspondence

Recall:

$$\mathrm{Sp}(2n,\mathbb{C})^L = \mathrm{SO}(2n+1,\mathbb{C}).$$

Corollary

Let $b \in B_{\mathrm{Sp}(2n,\mathbb{C})}(X) = B_{\mathrm{SO}(2n+1,\mathbb{C})}(X)$ of $\mathfrak{sl}(2)$-type. Then $\mathrm{Hit}^{-1}_{\mathrm{Sp}(2n,\mathbb{C})}(b)$ and $\mathrm{Hit}^{-1}_{\mathrm{SO}(2n+1,\mathbb{C})}(b)$ are related as follows:

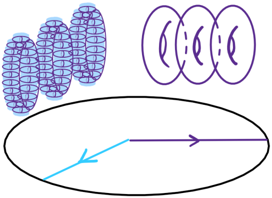
- *The abelian part of the spectral data are abelian torsors over dual abelian varieties.*
- *The non-abelian part of the spectral data are isomorphic.*

In example iii):

$$\begin{array}{ccc}
\mathbb{P}(1,1,2) & \overset{\cong}{\underset{\text{canon.}}{\longleftrightarrow}} & \mathbb{P}(1,1,2) \\
\downarrow & & \downarrow \\
\mathrm{Hit}^{-1}_{\mathrm{Sp}(2n,\mathbb{C})}(b_3) & & \mathrm{Hit}^{-1}_{\mathrm{SO}(2n+1,\mathbb{C})}(b_3) \\
\downarrow & & \downarrow \\
\mathrm{Prym}(\tilde{\Sigma}) & \overset{\text{duality}}{\longleftrightarrow} & \dfrac{\mathrm{Prym}(\tilde{\Sigma})}{\mathrm{Jac}(X)[2]}
\end{array}$$

App. 2: Solutions to decoupled Hitchin equation

Fredrickson, Mazzeo-Swoboda-Weiss-Witt:
Studied degeneration of solutions to Hitchin equation along rays to the ends of the moduli space $\mathcal{M}_{\mathrm{SL}(n,\mathbb{C})}(X)$:



Limits: hermitian metrics h_∞ singular at branch points of $\pi : \Sigma \to X$, such that

$$F_{\nabla^{h_\infty}} = 0, \qquad [\Phi, \Phi^{*h_\infty}] = 0. \tag{DHE}$$

Theorem

Let $(E, \Phi) \in \mathrm{Hit}_G^{-1}(b)$ *of* $\mathfrak{sl}(2)$*-type. There exits a solution to DHE determined by the semi-abelian spectral data.*

$$\begin{array}{ccccc}
\Sigma & \longleftarrow & \tilde{\Sigma} & \dashleftarrow & (L, u, h^{\mathrm{HE}}) \\
\downarrow \pi & \swarrow \tilde{\pi} & & \swarrow \tilde{\pi}^u_* & \\
X & \dashleftarrow & (E, \Phi, \tilde{\pi}^u_* h^{\mathrm{HE}}) & &
\end{array}$$

$\tilde{\pi}^u_*$: Hecke-modified pushforward

$(E, \Phi, \tilde{\pi}^u_* h^{\mathrm{HE}})$: solution to DHE

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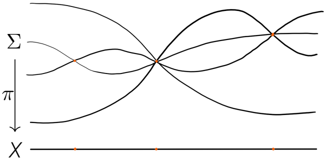

$$\begin{array}{ccc}
\mathbb{P}(1,1,2) & \xleftrightarrow[\text{canon.}]{\cong} & \mathbb{P}(1,1,2) \\
\downarrow & & \downarrow \\
\mathrm{Hit}^{-1}_{\mathrm{Sp}(2n,\mathbb{C})}(b_3) & & \mathrm{Hit}^{-1}_{\mathrm{SO}(2n+1,\mathbb{C})}(b_3) \\
\downarrow & & \downarrow \\
\mathrm{Prym}(\tilde{\Sigma}) & \xleftrightarrow{\text{duality}} & \dfrac{\mathrm{Prym}(\tilde{\Sigma})}{\mathrm{Jac}(X)[2]}
\end{array}$$

Thank you for your attention!

$$\begin{array}{ccccc}
\Sigma & \longleftarrow & \tilde{\Sigma} & \leftarrow - - - & (L, h^{\mathrm{HE}}) \\
\pi \downarrow & \swarrow \tilde{\pi} & & \swarrow \text{twisted } \tilde{\pi}_* & \\
X & \leftarrow - - - & (E, \Phi, \pi_*^{\mathrm{tw}} h^{\mathrm{HE}}) & &
\end{array}$$

solution to DHE